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On the Boolean Minimal Realization Problem in the Max-Plus Algebra: Addendum*

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In this addendum we present an upper bound for the minimal system order of a max-linear time-invariant DES that can be computed very efficiently, and we give some lemmas that characterize the *ultimate* behavior of the sequence $\left\{A^{\otimes^k}\right\}_{k=0}^{\infty}$ for a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$.

A Upper bounds for the minimal system order

Definition A.1 (Ultimately geometric impulse response [12, A4]) *Let* $\{G_k\}_{k=0}^{\infty}$ *be the impulse response of a max-linear time-invariant DES. If*

$$\exists k_0 \in \mathbb{N},\ \exists c \in \mathbb{N}_0,\ \exists \lambda \in \mathbb{R}_\varepsilon \text{ such that } \forall k \geqslant k_0 : G_{k+c} = \lambda^{\otimes^c} \otimes G_k \ , \tag{A.1}$$

then we say that the impulse response $\{G_k\}_{k=0}^{\infty}$ *is ultimately geometric.*

Note that an ultimately geometric sequence $G = \{G_k\}_{k=0}^{\infty}$ is also ultimately periodic. Furthermore, the smallest integers c and k_0 for which (A.1) holds, correspond to respectively the period of G and the length of the transient part of G.

Suppose that we have a DES that can be characterized by a triple (A, B, C). A sufficient but not necessary condition for the impulse response of this DES to be ultimately geometric is that A is irreducible (cf. Theorem 2.4). This will, e.g., be the case for a DES without separate independent subsystems, and with a cyclic behavior or with feedback from the output to the input (such as, e.g., a flexible production system in which the parts are carried around on a limited number of pallets that circulate in the system [3]).

Definition A.2 (Max-plus-algebraic weak column rank [11, 12]) *Let* $A \in \mathbb{R}_\varepsilon^{m\times n}$. *If* $A \neq \mathcal{E}_{m\times n}$ *then the max-plus-algebraic weak column rank of* A *is defined by*

$$\begin{aligned}\operatorname{rank}_{\oplus,\text{wc}}(A) = \min \Big\{ \#I \,\Big|\, & I \subseteq \{1,2,\ldots,n\} \text{ and } \forall k \in \{1,2,\ldots,n\}, \\ & \exists l \in \mathbb{N}_0,\ \exists i_1, i_2, \ldots, i_l \in I,\ \exists \alpha_1, \alpha_2, \ldots, \alpha_l \in \mathbb{R}_\varepsilon \\ & \text{such that } A_{.,k} = \bigoplus_{j=1}^{l} \alpha_j A_{.,i_j} \Big\} \ .\end{aligned}$$

By definition we have $\operatorname{rank}_{\oplus,\text{wc}}(\mathcal{E}) = 0$.

Efficient methods to compute the max-plus-algebraic weak column rank of a matrix are described in [4, 11, A2]. It is easy to verify that for any matrix $A \in \mathbb{R}_\varepsilon^{m\times n}$ we have $\operatorname{rank}_{\oplus,\text{Schein}}(A) \leqslant \operatorname{rank}_{\oplus,\text{wc}}(A)$.

Lemma A.3 *Let* G *be an ultimately geometric sequence with period* c. *Let* k_0 *be the length of the transient part of* G. *Then we have*

$$\operatorname{rank}_{\oplus,\text{wc}} H(G) = \operatorname{rank}_{\oplus,\text{wc}} \big(H(G)\big)_{\{1,2,\ldots,k\},\{1,2,\ldots,k\}} \qquad \text{for all } k \geqslant k_0 + c \ . \tag{A.2}$$

Proof: We shall prove this lemma for a sequence of numbers $g = \{g_k\}_{k=0}^{\infty}$. The extension of this proof to a sequence of matrices is straightforward.
Define $H_1 = \big(H(g)\big)_{.,\{1,2,\ldots,k_0+c\}}$ and $H_2 = \big(H(g)\big)_{\{1,2,\ldots,k_0+c\},\{1,2,\ldots,k_0+c\}}$.
First we show that $\operatorname{rank}_{\oplus,\mathrm{wc}} H(g) = \operatorname{rank}_{\oplus,\mathrm{wc}} H_1$.
Let $k \in \mathbb{N}$. We have

$$(H(G))_{.,k_0+k+1} = \begin{bmatrix} g_{k_0+k} \\ g_{k_0+k+1} \\ g_{k_0+k+2} \\ \vdots \end{bmatrix} .$$

Since g is ultimately geometric, there exists a number $\lambda \in \mathbb{R}_\varepsilon$ such that $g_{k_0+c+k} = \lambda^{\otimes^c} \otimes g_{k_0+k}$ for all $k \in \mathbb{N}$. Hence, $g_{k_0+rc+k} = \lambda^{\otimes^{rc}} \otimes g_{k_0+k}$ for all $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$, and thus also

$$\big(H(G)\big)_{.,k_0+rc+k+1} = \lambda^{\otimes^{rc}} \otimes \big(H(G)\big)_{.,k_0+k+1} \qquad \text{for all } r \in \mathbb{N}_0 \text{ and } k \in \mathbb{N} \ .$$

This implies that any column $\big(H(G)\big)_{.,k_0+c+l}$ with $l \in \mathbb{N}_0$ can be written as $\alpha \otimes \big(H(G)\big)_{.,k_0+s}$ for some $s \in \{1, 2, \ldots, c\}$ and some $\alpha \in \mathbb{R}_\varepsilon$. As a consequence, we have

$$\operatorname{rank}_{\oplus,\mathrm{wc}} H(G) = \operatorname{rank}_{\oplus,\mathrm{wc}} \big(H(G)\big)_{.,\{1,2,\ldots,k_0+c\}} = \operatorname{rank}_{\oplus,\mathrm{wc}} H_1 \ .$$

Using a similar reasoning as the one that has been used above, it can be shown that any row $\big(H_1\big)_{k_0+c+l,.}$ with $l \in \mathbb{N}_0$ can be written as $\alpha \otimes \big(H_1\big)_{k_0+s,.}$ for some $s \in \{1, 2, \ldots, c\}$ and some $\alpha \in \mathbb{R}_\varepsilon$. So if we have

$$(H_2)_{.,k} = \bigoplus_{j=1}^{l} \alpha_j (H_2)_{.,i_j}$$

for some $l, k, i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, k_0 + c\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_l \in \mathbb{R}_\varepsilon$, then we also have

$$(H_1)_{.,k} = \bigoplus_{j=1}^{l} \alpha_j (H_1)_{.,i_j} \ .$$

This implies that $\operatorname{rank}_{\oplus} H_1 = \operatorname{rank}_{\oplus,\mathrm{wc}} (H_1)_{\{1,2,\ldots,k_0+c\},.} = \operatorname{rank}_{\oplus,\mathrm{wc}} H_2$.
Hence, $\operatorname{rank}_{\oplus,\mathrm{wc}} H(G) = \operatorname{rank}_{\oplus,\mathrm{wc}} H_2$. As a consequence, (A.2) holds. □

Remark A.4 Note that Lemma A.3 implies that if G is an ultimately geometric sequence then $\operatorname{rank}_{\oplus,\mathrm{wc}} H(G)$ is finite and can be determined using a finite number of elementary operations.

The max-plus-algebraic sum of sequences is defined as follows. If $G = \{G_k\}_{k=0}^{\infty}$ and $H = \{H_k\}_{k=0}^{\infty}$ with $G_k, H_k \in \mathbb{R}_\varepsilon^{l\times m}$ for all $k \in \mathbb{N}$, then $G \oplus H$ is a sequence with $(G \oplus H)_k = G_k \oplus H_k$ for all $k \in \mathbb{N}$.
From Theorem 3.1 it follows that the impulse response of a max-linear time-invariant DES can always be considered as the max-plus-algebraic sum of a finite number of ultimately geometric impulse responses (see also [1, 11, 12]).

Theorem A.5 *Let g be the impulse response of a max-linear time-invariant SISO DES with $g \neq \{\varepsilon\}_{k=0}^{\infty}$. Let g_1, g_2, …, g_s be ultimately geometric sequences such that $g = g_1 \oplus g_2 \oplus \cdots \oplus g_s$. Then there exists a state space realization of g of order $\sum_{i=1}^{s} \operatorname{rank}_{\oplus,\mathrm{wc}} \big(H(g_i)\big)$.*

Proof: See [11, 12]. $\square$

Proposition A.6 *For any ultimately periodic sequence G we can compute a finite upper bound for the minimal system order of the max-linear time-invariant DES the impulse response of which coincides with G using a finite number of elementary operations.*

Proof: This is a direct consequence of Lemma A.3 and Theorem A.5. $\square$

B The ultimate behavior of the sequence of consecutive max-plus-algebraic matrix powers

If we permute the rows or the columns of the max-plus-algebraic identity matrix, we obtain a max-plus-algebraic permutation matrix. If $P \in \mathbb{R}_\varepsilon^{n\times n}$ is a max-plus-algebraic permutation matrix, then we have $P \otimes P^T = P^T \otimes P = E_n$. A matrix $R \in \mathbb{R}_\varepsilon^{m\times n}$ is a max-plus-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i, j with $i > j$.

Lemma B.1 *If $A \in \mathbb{R}_\varepsilon^{n\times n}$ then there exists a max-plus-algebraic permutation matrix $P \in \mathbb{R}_\varepsilon^{n\times n}$ such that the matrix $\hat{A} = P \otimes A \otimes P^T$ is a max-plus-algebraic block upper triangular matrix of the form*

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \ldots & \hat{A}_{1l} \\ \mathcal{E} & \hat{A}_{22} & \ldots & \hat{A}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \ldots & \hat{A}_{ll} \end{bmatrix} \tag{A.3}$$

with $l \geqslant 1$ and where the matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are square and irreducible. The matrices $\hat{A}_{11}, \hat{A}_{22}, \ldots, \hat{A}_{ll}$ are uniquely determined to within simultaneous permutation of their rows and columns, but their ordering in (A.3) is not necessarily unique.

Proof: See, e.g., [1]. This lemma is also the max-plus-algebraic equivalent of a result of [A5]. A proof of the uniqueness assertion can be found in [A1] (Theorem 3.2.4[1]). $\square$

The form in (A.3) is called the max-plus-algebraic Frobenius normal form of the matrix A. Note that if A is irreducible then there is only one block in (A.3) and then A is a max-plus-algebraic Frobenius normal form of itself.
Let $A \in \mathbb{B}^{n\times n}$ (or $A \in \mathbb{R}_\varepsilon^{n\times n}$). If $\hat{A} = P \otimes A \otimes P^T$ is the max-plus-algebraic Frobenius normal form of A, then we have $A = P^T \otimes \hat{A} \otimes P$. Hence,

$$A^{\otimes^k} = \left(P^T \otimes \hat{A} \otimes P\right)^{\otimes^k} = P^T \otimes \hat{A}^{\otimes^k} \otimes P$$

for all $k \in \mathbb{N}$. Therefore, we may consider without loss of generality the sequence $\left\{\hat{A}^{\otimes^k}\right\}_{k=0}^\infty$ instead of the sequence $\left\{A^{\otimes^k}\right\}_{k=0}^\infty$. Furthermore, since the transformation from A to $\hat{A}$ corresponds to a simultaneous reordering of the rows and columns of A (or to a reordering of the vertices of $\mathcal{G}(A)$), we have $c(A) = c(\hat{A})$.

The following lemma is an extension of Theorem 2.4 and a corrected version of a lemma that can be found in [A6]:

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[1] Although this theorem is stated for $(0,1)$-matrices, there is a one-to-one correspondence between a max-plus-algebraic boolean matrix and a $(0,1)$-matrix if we let 0 and ε correspond with 1 and 0 respectively.

Lemma B.2 *Let* $\hat{A} \in \mathbb{R}_\varepsilon^{n\times n}$ *be a matrix of the form (A.3) where the matrices* $\hat{A}_{11}$, ..., $\hat{A}_{ll}$ *are square and irreducible. Let* λ_i *and* c_i *be respectively the max-plus-algebraic eigenvalue and the cyclicity of* $\hat{A}_{ii}$ *for* $i = 1, \ldots, l$. *Define sets* α_1, ..., α_l *such that* $\hat{A}_{\alpha_i\alpha_j} = \hat{A}_{ij}$ *for all* i, j *with* $i \leqslant j$.
Define

$$S_{ij} = \Big\{\, \{i_0, \ldots, i_s\} \subseteq \{1, \ldots, l\} \,\Big|\, i = i_0 < i_1 < \ldots < i_s = j \text{ and } \hat{A}_{i_r i_{r+1}} \neq \mathcal{E} \text{ for } r = 0, \ldots, s-1 \,\Big\}$$

$$\Gamma_{ij} = \bigcup_{\gamma \in S_{ij}} \gamma$$

$$\Lambda_{ij} = \begin{cases} \{\lambda_t | t \in \Gamma_{ij}\} & \textit{if } \Gamma_{ij} \neq \emptyset\,, \\ \{\varepsilon\} & \textit{if } \Gamma_{ij} = \emptyset\,, \end{cases}$$

$$c_{ij} = \begin{cases} \operatorname{lcm}\{\, c_t \,|\, t \in \Gamma_{ij} \,\} & \textit{if } \Gamma_{ij} \neq \emptyset \textit{ and } c_t \neq 0 \textit{ for some } t \in \Gamma_{ij}\,, \\ 1 & \textit{otherwise}\,, \end{cases}$$

for all i, j *with* $i < j$. *We have*

$$\forall i, j \in \{1, \ldots, l\} \textit{ with } i > j : \left(\hat{A}^{\otimes^k}\right)_{\alpha_i\alpha_j} = \mathcal{E}_{n_i \times n_j} \quad \textit{for all } k \in \mathbb{N}\,. \tag{A.4}$$

Moreover, there exists an integer $K \in \mathbb{N}$ *such that*

$$\forall i \in \{1, \ldots, l\} : \left(\hat{A}^{\otimes^{k+c_i}}\right)_{\alpha_i\alpha_i} = \lambda_i^{\otimes^{c_i}} \otimes \left(\hat{A}^{\otimes^k}\right)_{\alpha_i\alpha_i} \quad \textit{for all } k \geqslant K \tag{A.5}$$

and

$$\forall i, j \in \{1, \ldots, l\} \textit{ with } i < j, \forall p \in \alpha_i, \forall q \in \alpha_j, \exists \gamma_0, \ldots, \gamma_{c_{ij}-1} \in \Lambda_{ij} \textit{ such that}$$
$$\left(\hat{A}^{\otimes^{kc_{ij}+c_{ij}+s}}\right)_{pq} = {\gamma_s}^{\otimes^{c_{ij}}} \otimes \left(\hat{A}^{\otimes^{kc_{ij}+s}}\right)_{pq} \quad \textit{for all } k \geqslant K \textit{ and for } s = 0, \ldots, c_{ij}-1\,. \tag{A.6}$$

Furthermore, for each combination i, j, p, q *with* $i < j$, $p \in \alpha_i$ *and* $q \in \alpha_j$, *there exists at least one index* $s \in \{0, \ldots, c_{ij}-1\}$ *such that the smallest* γ_s *for which (A.6) holds is equal to* $\max \Lambda_{ij}$.

Proof: See [A3]. □

If $G = \{G_k\}_{k=0}^{\infty}$ is the impulse response of a max-linear time-invariant DES and if the triple (A, B, C) is a state space realization of the DES, then it follows from Lemmas B.1 and B.2 that the period of G is a divisor of the cyclicity $c(A)$ of the system matrix A.

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