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Robustness Analysis of Platoon Control for Mixed Types of Vehicles

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***Abstract*—Currently, with the development of driving technologies, driverless vehicles gradually are becoming more and more available. Therefore, there would be a long period of time during which self-driving vehicles and human-driven vehicles coexist. However, for a mixed platoon, it is hard to control the formation due to the existence of the manual vehicles resulting in weak robustness and slow consensus rate on this system of platoons because of uncertainties caused by human factors for manual vehicles. In order to solve this problem, we establish models of mixed platoons with mixed types of connected and automated vehicles (CAVs), human-driven vehicles (HDVs) and HDVs without the vehicle awareness device (HDVWs). We subsequently design $\mathcal{H}_\infty$ controllers for the mixed platoons to realize the formation consensus. In addition, we use the $\mathcal{H}_\infty$ norm of mixed platoons as the control objective investigating the robustness of the control algorithms in alleviating the platoon uncertainties. Furthermore, conditions are proved to maintain the stability of the mixed platoons, and the stability is analyzed based on the variation of the penetration rate of the manual vehicles. Finally, we formulate conditions for parameters according to the definition of string stability to avoid the collisions of vehicles. The results in this study are tested with simulations and suggest that the presented controllers can ensure the consensus of mixed platoons under uncertainties.**

***Index Terms*—Traffic Network Control, Mixed Traffic Flow, $\mathcal{H}_\infty$ Control.**

I. Introduction

WITH the development of urbanization and the popularization of vehicles, the number of motor vehicles is continuing to increase. Meanwhile, three major problems in the field of public transport including traffic safety, congestion and pollution, are becoming increasingly serious. In order to address these problems, automobiles are made to be electric, intelligent, network-connected and shared. With the innovation of computer control technologies, more and more automatic control technologies have been applied to automobiles. Driverless cars emerged with the demands of security, environmental-friendliness and higher cost performance. In other words, driverless cars will be popular in the future. But there will be a transition stage between manned and driverless vehicles [1].

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A platoon is a group of vehicles in close proximity that improves the efficiency of transportation. Based on platoon driving pattern, driving experience can become safer and more comfortable [2]. Due to this is a special stage, we focus on heterogeneous vehicular platoon, i.e., mixed platoon. For example, the scenarios in [3]–[8] described the mixed platoons including two types of vehicles: CAVs and HDVs. However, the third vehicle, i.e., HDVWs are neglected in these scenarios.

The goal of the platoons is to achieve consensus which means every vehicle drives at the same velocity and an expected position. For this control, controllers were established to coordinate the vehicles in platoons by integrating the feedback of neighborhood information in [9]. It is found that the control frameworks of platoons can be classified into centralized control and distributed control [2], [10]–[15]. The former is more complex in computation. As such, it may not be suitable for large vehicle platoons. However, the distributed control, such as adaptive sliding model control [11] and distributed model predictive control (DMPC) [12], designs individual controllers for each vehicles, and realizes the platoon coordination through information exchange among vehicles.

In addition, robustness and string stability are the two crucial points during designing the platoon control protocols [16], [17]. In view of the changes in working conditions, external interference and modeling errors, it is difficult to model actual platoons accurately, and various failures of the system will also increase uncertainties of the models. In platoons, uncertainties bring instability to other vehicles. How to design a controller to make the uncertain objects meet the control quality requirements to achieve the robust control, has become a key research topic for researchers [3], [18]. In [19], the authors utilized the neural network to tackle uncertain vehicle dynamics. Unfortunately, this method is not suitable for the linear systems. Among the reported controllers, $\mathcal{H}_\infty$ control is more commonly used [20]. $\mathcal{H}_\infty$ control is used in the linear oscillation system in [21], because it almost meets the L_2 string stability condition and can provide a choice between vehicle following performance and string stability.

When designing controllers for platoons, another particular difficulty is known as string instability [22]. A platoon is considered as string instable, if small disturbances within the platoon amplify and cause a traffic jam in the end. To solve this problem, the property of string stability has been widely studied. Ever since the original definition of string was proposed, a number of definitions for string stability was given, such as strong frequency-domain string stability [23], input-to-state string stability [24], $\mathcal{L}_p$ string stability [25], [26], head-to-tail stability (HTS) [27], [28] and so on. In particular, HTS was introduced in mixed traffic [29], by which the vehicles in a platoon travel with the same velocity and maintain constant headways when the platoon system is HTS. The platoon system is HTS, if the function of outputs, denoted as G, satisfies $\parallel G \parallel_{\mathcal{H}_\infty} \leq 1$.

As mentioned above, there will be a long period of time for mixed platoons, that is, the coexistence of human-driven and driverless vehicles [30] until the human-driven vehicles are completely replaced by the driverless vehicles. However, a few researchers consider the third vehicles. This type of vehicles can exchange information with other vehicles and roadside controllers like the autonomous vehicles. Furthermore, their behaviors are controlled by the drivers instead of driving automatically. During the transition phase, an important issue is how to deal with various kinds of vehicles to get different vehicles coordinate with each other, and how to improve the driving efficiency of the traffic networks.

In this paper, we establish a mathematical model for mixed platoons composed of different kinds of vehicles, including CAVs, HDVs and HDVWs. This model called formation consensus, which is suitable for avoiding inter-vehicular collisions [31]. Based on this model, we design control laws to adapt the velocity difference and the position difference between adjacent vehicles to achieve smooth and efficient platoon driving. Moreover, we propose a method to quantify the control effects of platoons against the platoon uncertainties with an $\mathcal{H}_\infty$ norm based on graph-theoretic notions and Vehicle to Vehicle (V2V) technology [32]. In addition, the effect of manual-driving vehicles on the robustness of mixed platoon control is analyzed. Although the above methods ensure the consistency of vehicle behavior, the distance between vehicles are becoming increased because of the existence of the HDVWs. To solve this problem, the conditions for HTS are deduced.

The paper is organized as follows. We start by introducing our notations in Section II, and in Section III, the model of mixed platoons of formation consensus is presented. In Section IV, bounds on the robustness metrics of the models above are analyzed. The results are tested by simulation in Section V. Finally, the simulation results and conclusions are summarized in Section VI.

II. Notations and definitions

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real and natural numbers, respectively. The topology of a multi-vehicle system can be characterized by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with node set $\mathcal{V} = \{\text{ve}_1, \text{ve}_2, \dots, \text{ve}_N\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ satisfies $a_{ij} > 0$ if and only if $(\text{ve}_i, \text{ve}_j) \in \mathcal{E}$. Otherwise, $a_{ij} = 0$. The weighted degree matrix of a graph $\mathcal{G}$ is therefore denoted by $D = diag\{d_1, d_2, \dots, d_n\} \in \mathbb{R}^{n \times n}$. The degree of node ve_i is denoted by $d_i = \sum_{j=1}^{n} A_{ij}$. The Laplacian matrix of the graph is given by $L = D - A$.

In this article, all of the networks are analyzed as undirected graphs. Then the order of nodes in $\mathcal{E}$ is irrelevant and the corresponding adjacency matrix is symmetric. The set of neighbors of agent i is defined by $\mathcal{N}_i = \{\text{ve}_j \in \mathcal{V} | (\text{ve}_j, \text{ve}_i) \in \mathcal{E}\}$. For a given set of nodes $S \subset V$, the ground Laplacian induced by S is given by L_g [33], which is obtained by removing the rows and columns of L corresponding to the nodes (namely grounded nodes) of S. In this paper, the grounded nodes represent reference vehicles. For a given set of nodes I, the number of cardinality of the set is denoted by $|I|$ (which we call cardinal number). A^* is the conjugate transpose matrix of A. The symbol $\otimes$ denotes Kronecker product [34].

III. Problem statement

In this paper, we consider a network consisting of N vehicles, where the network corresponds to a platoon with a number of vehicles. In a platoon, each vehicle $v_i \in \mathcal{V}$ is either a follower vehicle $v_i \in \mathcal{F}$ or a reference vehicle $v_i \in \mathcal{R}$. A reference vehicle is the vehicle that can receive the control signals and communicate with some of the other vehicles, and the followers are the vehicles that are receiving the signals from the reference vehicles. In particular, a mixed platoon may include three kinds of vehicles, i.e. CAVs, HDVs, and HDVWs. In detail, CAVs could not only communicate information with other vehicles, but they can also make the driving decisions automatically. HDVs can only transmit information with others, but they cannot drive automatically. HDVWs are the normal vehicles that cannot be controlled automatically, which can be considered as disturbances in a platoon due to the uncertainty of the behavior of the HDVWs and the number of HDVWs. Compared with CAVs and HDVs, the HDVWs can neither be detected through information exchange, nor be controlled by the order given by the leader of the platoon or the traffic center. Due to this reason, the HDVWs is undetectable and uncontrollable in mixed vehicle platoons. Therefore, uncertainties are introduced into mixed vehicle platoons.

Normally, during manual driving, if the speed of a front car changes, the driver of the rear car needs to observe the variation of the distance between the front car and the rear car by eyes. According to the distance between the front car and rear car judged by the driver, the speed and acceleration are adapted manually. But, for human drivers, there is a long reaction time compared to the CAVs. So a large safe distance should be maintained from the front car for the consideration of driving safety. Even with the help of intelligent auxiliary systems, the decision maker is still human in the end, which means a certain reaction time is required. In a platoon, if all the vehicles can drive with the same speed and coordinate with each other, then the distance between vehicles can be reduced, and the traffic capacity and transport efficiency on a road can be improved [35]. However, in this paper, we consider a mixed platoon with multiple vehicles, including CAVs, HDVs, and HDVWs. For a mixed platoon, the dynamics of HDVWs is uncertain factor that are hard to predict and control. In such a circumstance, we build a platoon model taking the HDVWs as disturbances, and we investigate the control method aiming at

TABLE I
VEHICULAR PARAMETERS OF PLATOON FORMATION CONSENSUS

Variable	Description
q_{ve_i}	gain of position of the ith follower-vehicle
$q_{\text{ve}_i-\text{RC}}$	gain of position of the ith follower-vehicle if the sender of information of this follower-vehicle is a CAV/HDV or a reference vehicle
$q_{\text{ve}_i-\text{HDVW}}$	gain of position of the ith follower-vehicle if the sender of information of this follower-vehicle is an HDVW
d_{ve_i}	gain of velocity of the ith follower-vehicle
$d_{\text{ve}_i-\text{RC}}$	gain of velocity of the ith follower if the sender of information of this follower-vehicle is a CAV/HDV or reference vehicle
$d_{\text{ve}_i-\text{HDVW}}$	gain of velocity of the ith follower-vehicle if the former of this follower-vehicle is an HDVW
q_{HDVW}	gain of position if the former of this follower-vehicle is an HDVW
d_{HDVW}	gain of velocity if the former of this follower-vehicle is an HDVW

achieving formation consensus under the coexistence of these uncertainties. Therefore, this paper will consider controlling the vehicle speed and position in a platoon to make the vehicles maintain a constant speed and constant distance in such an uncertain environment. Before describing the models, some assumptions are clarified: Because we focus on consensus control for a mixed platoon, lane change behavior is prohibited. In addition, in view of the rapid development of modern communication technology, communications are assumed to be perfect.

A. Basic Model

The dynamics of the reference vehicles are given by

$$\dot{v}_0(t) = 0, \tag{1}$$

where v_0 is the velocity of reference vehicles in the platoon. In order to improve the traffic flow, v_0 is given as a control command by upper controllers such as roadside controllers.

The dynamics of each follower vehicles are governed by

$$\dot{v}_i(t) = a_i(t), \tag{2}$$

where $a_i(t)$ and $v_i(t)$ are the acceleration and velocity of the ith follower-vehicle.

B. The Models of Different Types of Vehicles

In order to control the vehicles in a platoon to achieve the constant speed and constant distance in the platoon, formation consensus, i.e. in which vehicles finally have not only the same velocity but also the same distance between front vehicles and rear vehicles, will be explored in this paper. Thus, in this subsection, we will consider different models for this cases. To facilitate the presentation of the proposed model, all of the vehicular parameters throughout this paper are listed in Table I.

In this scenario, the goal of all of the followers is to reach not only the expected speed but also the same distance between neighbors. The acceleration dynamics of follower-vehicles are as follows:

$$\begin{aligned}\dot{v}_i(t) = a_i(t) = &\sum_{j\in\mathcal{N}_i}\left[q_{\text{ve}_i}\left(p_j(t) - p_i(t) + \Delta_{ij}\right)\right] \\ &+ \sum_{j\in\mathcal{N}_i}\left[d_{\text{ve}_i}\left(v_j(t) - v_i(t)\right)\right],\end{aligned} \tag{3}$$

where $v_i(t)$ and $p_i(t)$ are the velocity and position of the ith follower-vehicle, Δ_{ij} is a specific constant distance between the ith follower vehicle and the jth follower vehicle, which should satisfy $\Delta_{ij} = p_i^* - p_j^*$ where p_i^* is the desired position of the ith follower-vehicle. Moreover, q_{ve_i} is the gain of position of the ith follower-vehicle, and d_{ve_i} is a gain of velocity of the ith follower-vehicle. Likewise, refining the formula (3) yields the following:

$$\begin{aligned}\dot{v}_i(t) = &\sum_{j_1\in\mathcal{R}_i}\left[q_{\text{ve}_i-\text{RC}}\left(p_{j_1}(t) - p_i(t) + \Delta_{ij_1}\right)\right] \\ &+ \sum_{j_1\in\mathcal{R}_i}\left[d_{\text{ve}_i-\text{RC}}\left(v_{j_1}(t) - v_i(t)\right)\right] \\ &+ \sum_{j_2\in\mathcal{H}_i}\left[q_{\text{ve}_i-\text{HDVW}}\left(e_{j_2}(t) - p_i(t) + \Delta_{ij_2}\right)\right] \\ &+ \sum_{j_2\in\mathcal{H}_i}\left[d_{\text{ve}_i-\text{HDVW}}\left(\dot{e}_{j_2}(t) - v_i(t)\right)\right] \\ &+ \sum_{j_3\in\mathcal{C}_i}\left[q_{\text{ve}_i-\text{RC}}\left(p_{j_3}(t) - p_i(t) + \Delta_{ij_3}\right)\right] \\ &+ \sum_{j_3\in\mathcal{C}_i}\left[d_{\text{ve}_i-\text{RC}}\left(v_{j_3}(t) - v_i(t)\right)\right],\end{aligned} \tag{4}$$

where e_j represents the position of the HDVWs. The gains of position ($q_{\text{ve}_i-\text{RC}}$, $q_{\text{ve}_i-\text{HDVW}}$) and the gains of velocity ($d_{\text{ve}_i-\text{RC}}$, $d_{\text{ve}_i-\text{HDVW}}$) in (4) are listed in Table I.

C. The Model of Mixed Platoons

In the previous subsection, we presented the acceleration model for each vehicle. In this subsection, the acceleration model of vehicles will be applied to a model of a platoon. Before modeling, we need some explanations for the HDVWs. HDVWs are distinctive compared to CAVs and HDVs. The reasons are as follows: On the one hand, HDVWs are hard to be affected by other vehicle through information exchange; on the other hand, the existence of HDVWs has uncertainty in dynamics of HDVWs in the platoon. Accordingly, HDVWs are considered as the disturbances in platoons.

In this subsection, we will focus on models of mixed platoons, which have three kinds of vehicles, i.e. CAVs, HDVs and HDVWs, for the formation consensus.

Position errors are defined compared with the desired position as $\tilde{p}_i = p_i - p_i^*$. (4) is rewritten as follows:

$$\dot{\tilde{\mathbf{x}}}(t) = B\tilde{\mathbf{x}}(t) + Fr_{\text{HDVW}}\tilde{\mathbf{E}}(t), \tag{5}$$

where

$$\tilde{\mathbf{x}}(t) = [\tilde{p}_1(t)\ \ \tilde{p}_2(t)\ \ \ldots\ \ \tilde{p}_n(t)\ \ \dot{\tilde{p}}_1(t)\ \ \dot{\tilde{p}}_2(t)\ \ \ldots\ \ \dot{\tilde{p}}_n(t)]^T,$$

$$\tilde{\mathbf{E}}(t) = [\tilde{\mathbf{e}}(t)\ \dot{\tilde{\mathbf{e}}}(t)]^T,$$

$$B = B_1 + B_2,$$

$$F = \begin{bmatrix} 0_{|\mathcal{F}|\times 1} & 0_{|\mathcal{F}|\times 1} \\ \bar{I}_j & \bar{I}_j \end{bmatrix},$$

$$r_{\text{HDVW}} = \begin{bmatrix} d_{\text{ve}_i-\text{HDVW}} & \\ & q_{\text{ve}_i-\text{HDVW}} \end{bmatrix},$$

which

$$B_1 = \begin{bmatrix} 0_{|\mathcal{F}|\times|\mathcal{F}|} & I_{|\mathcal{F}|\times|\mathcal{F}|} \\ 0_{|\mathcal{F}\times|\mathcal{F}|} & 0_{|\mathcal{F}|\times|\mathcal{F}|} \end{bmatrix},$$

$$B_2 = -\begin{bmatrix} 0_{|\mathcal{F}|\times|\mathcal{F}|} & I_{|\mathcal{F}|\times|\mathcal{F}|} \\ B_{21} & B_{22} \end{bmatrix},$$

$$B_{21} = \mathbf{diag}(q_{\text{ve}_i}, \ldots, q_{\text{ve}_n}) \times L_g,$$

$$B_{22} = \mathbf{diag}(d_{\text{ve}_i}, \ldots, d_{\text{ve}_n}) \times L_g.$$

Note that $\bar{I}_j$ is a matrix in which each element is either 0 or 1, and the value of 0 or 1 corresponds to the positions of the HDVWs: When the front vehicle of the ith follower-vehicle is an HDVW, then the element value is 1; when a front vehicle of the ith vehicle is a CAV or an HDV, then the element value is 0.

IV. Robustness of platoon formation consensus

In this section, we provide a condition to evaluate the robustness of the network formation dynamics.

The transfer function of (5) is:

$$G_{\text{f}}(s) = (s^2 I + B_{22}s + B_{21})^{-1}(d_{\text{HDVW}}\ \bar{I}_j\ s + q_{\text{HDVW}}\ \bar{I}_j); \quad (6)$$

see the Appendix A for the details.

Note that the column vector $\bar{I}_j$ contains two elements that are either 0 or 1. Actually, the bth element in the transfer function matrix is the sum of the elements of the inverse matrix in the bth row and certain columns. The certain columns equal to rows to which nonzero element located in vector $\bar{I}_j$.

The presence of HDVWs leads to the uncertainty of the platoon, because HDVWs neither send driving information to other vehicles nor receive driving information from the reference vehicles and the front vehicle. Thus, it is indispensable to estimate the impact of HDVWs on the system state (i.e. platoon speed and position). Accordingly, the $\mathcal{H}_\infty$-norm of the defined transfer function is adopted as the control performance. Suppose the certain relationship between two coefficients (d_{HDVW}, the velocity coefficient of the HDVWs and q_{HDVW}, the position coefficient of the HDVWs) is as follows:

$$d_{\text{HDVW}} = a_1 q_{\text{HDVW}}, \quad (7)$$

where a_1 is a constant value; in this paper we choose $a_1 = 1$ since the drivers of the HDVWs have the same level of response ratio to the position error and the velocity error.

Let

$$\mathbf{D}_q = \mathbf{diag}(q_{\text{ve}_i}, \ldots, q_{\text{ve}_n}),$$

$$\mathbf{D}_d = \mathbf{diag}(d_{\text{ve}_i}, \ldots, d_{\text{ve}_n}),$$

then

$$(s^2 I + B_{22}s + B_{21})^{-1} = (s^2 I + \mathbf{D}_d L_g s + \mathbf{D}_q L_g)^{-1}. \quad (8)$$

The subsequent analysis is based on the premise that there are two kinds of vehicles in platoon, i.e. CAVs and HDVs are considered as C vehicles. Then the following two matrices are constant matrices:

$$\mathbf{D}_q = q_{\text{c}} \otimes \mathbf{I},$$

$$\mathbf{D}_d = d_{\text{c}} \otimes \mathbf{I}.$$

By rewriting (8) we have

$$(s^2 I + B_{22}s + B_{21})^{-1} = (s^2 I + q_{\text{c}} L_g s + d_{\text{c}} L_g)^{-1}, \quad (9)$$

and (9) can be put in diagonal form as:

$$\begin{aligned}(s^2 I + B_{22}s + B_{21})^{-1} &= (s^2 I + d_{\text{c}} L_g s + q_{\text{c}} L_g)^{-1} \\ &= Q(s^2 I + (d_{\text{c}}s + q_{\text{c}})\Lambda)^{-1} Q^T,\end{aligned} \quad (10)$$

where Q is a matrix formed by the eigenvectors of L_g, and Λ is a diagonal matrix in which the elements are as follows:

$$G_{\text{f}i}(s) = \frac{1}{s^2 + d_{\text{c}}\lambda_i(L_g)s + q_{\text{c}}\lambda_i(L_g)}, i \in \{1, 2, \ldots, |\mathcal{C}|\}. \quad (11)$$

where $|\mathcal{C}|$ is the total number of CAVs and HDVs. Note that parameters d_{C} and q_{C} are proportional to each other, i.e. $d_{\text{C}} = \text{C}q_{\text{C}}$. When the variation of vehicle speed is more sensitive than the variation of vehicle location, C is larger than 1.

Theorem 1. *For a mixed platoon that contains three kinds of vehicles, i.e. CAVs, HDVs and HDVWs, the velocities and positions error of all the vehicles in the platoon will converge to the reference velocity and the reference positions error under the control law in (5), and the platoon system is robust to the uncertainties (i.e. the HDVWs), if the following condition is satisfied.*

$$\frac{d_{\text{HDVW}}}{D} \leq 1, \quad (12)$$

where

$$D = 2\sqrt{-\frac{D_1}{d_{\text{C}}^2} + D_2 + 1} + \frac{D_1}{d_{\text{C}}^2 - q_{\text{C}}^2} - D_2 - 2,$$

$$D_1 = \frac{9q_{\text{C}}^4}{d_{\text{C}}^2(d_{\text{C}}^2 - q_{\text{C}}^2)},$$

$$D_2 = \frac{6q_{\text{C}}^3}{d_{\text{C}}^2(d_{\text{C}}^2 - q_{\text{C}}^2)},$$

and $C^2 q_{\text{C}} > \frac{3}{2}$, $d_{\text{C}} = Cq_{\text{C}}$.

Proof: From (6) and (7), the transfer function of (5) can be rewritten as follows:

$$G_{\text{f}}(s) = d_{\text{HDVW}}(s^2 I + B_{22}s + B_{21})^{-1}(\bar{I}_j\ s + \bar{I}_j). \quad (13)$$

The subsystem of G_{f} is not disturbed by HDVWs if the element in $\bar{I}_j$ is 0. In order to explore robustness, we pay more attention to the case where the element in $\bar{I}_j$ is 1.

The amplitude of (13) satisfies:

$$|G_{\mathrm{f}}(j\omega)|^2 = \frac{d_{\mathrm{HDVW}}^2(\omega^2+1)}{\omega^4 + (d_{\mathrm{c}}^2\lambda_i^2 - 2q_{\mathrm{c}}\lambda_i)\omega^2 + q_{\mathrm{c}}^2\lambda_i^2}.$$

Let $x = \omega^2$, $g(x) = \frac{1}{|G_{\mathrm{f}}(j\omega)|^2}$. More exactly,

$$\begin{aligned} g(x) &= \frac{x+1}{d_{\mathrm{HDVW}}^2} + \frac{q_{\mathrm{c}}^2\lambda_i^2 - d_{\mathrm{c}}^2\lambda_i^2 + 2q_{\mathrm{c}}\lambda_i + 1}{d_{\mathrm{HDVW}}^2(x+1)} + \frac{d_{\mathrm{c}}^2\lambda_i^2 - 2q_{\mathrm{c}}\lambda_i - 2}{d_{\mathrm{HDVW}}^2} \\ &\geq 2\sqrt{\frac{q_{\mathrm{C}}^2\lambda_i^2 - d_{\mathrm{C}}^2\lambda_i^2 + 2q_{\mathrm{C}}\lambda_i + 1}{d_{\mathrm{HDVW}}^4}} + \frac{d_{\mathrm{C}}\lambda_i^2 - 2q_{\mathrm{C}}\lambda_i - 2}{d_{\mathrm{HDVW}}^2} \\ &= \frac{2\sqrt{q_{\mathrm{C}}^2\lambda_i^2 - d_{\mathrm{C}}^2\lambda_i^2 + 2q_{\mathrm{C}}\lambda_i + 1} + d_{\mathrm{C}}\lambda_i^2 - 2q_{\mathrm{C}}\lambda_i - 2}{d_{\mathrm{HDVW}}^2}. \end{aligned} \quad (14)$$

In order to obtain the minimum value of (14), λ_i is considered as an independent variable.

Let

$$h(\lambda_i) = 2\sqrt{q_{\mathrm{C}}^2\lambda_i^2 - d_{\mathrm{C}}^2\lambda_i^2 + 2q_{\mathrm{C}}\lambda_i + 1} + d_{\mathrm{C}}\lambda_i^2 - 2q_{\mathrm{C}}\lambda_i - 2, \quad (15)$$

and then take the derivative of this equation as follows:

$$h'(\lambda_i) = \frac{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i + 2q_{\mathrm{C}}}{\sqrt{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^2 + 2q_{\mathrm{C}}\lambda_i + 1}} + d_{\mathrm{C}}^2\lambda_i - q_{\mathrm{C}}. \quad (16)$$

Then we make (16) equal to 0. We can acquire a solution by solving this equation as follows:

$$\lambda_i^* = \frac{3q_{\mathrm{C}}^2}{d_{\mathrm{C}}^2(d_{\mathrm{C}}^2 - q_{\mathrm{C}}^2)}. \quad (17)$$

$\lambda_i = \lambda_i^*$ may be an extreme point of $h(\lambda_i)$. To further confirm whether it is an extreme point, the secondary derivative of $h(\lambda_i)$ is calculated as follows:

$$h^{''}(\lambda_i) = \frac{-d_{\mathrm{C}}^2}{(\sqrt{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^2 + 2q_{\mathrm{C}}\lambda_i + 1})^3} + d_{\mathrm{C}}^2. \quad (18)$$

Therefore,

$$\begin{aligned} h^{''}(\lambda_i^*) &= \frac{-d_{\mathrm{C}}^2}{(\sqrt{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^{*2} + 2q_{\mathrm{C}}\lambda_i^* + 1})^3} + d_{\mathrm{C}}^2 \\ &= d_{\mathrm{C}}^2 \cdot \frac{(\sqrt{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^{*2} + 2q_{\mathrm{C}}\lambda_i^* + 1})^3 - 1}{(\sqrt{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^{*2} + 2q_{\mathrm{C}}\lambda_i^* + 1})^3}, \end{aligned} \quad (19)$$

where $d_{\mathrm{C}}^2 > 0$, $(\sqrt{(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^{*2} + 2q_{\mathrm{C}}\lambda_i^* + 1})^3 > 0$.

Accordingly, if $\mathrm{C}^2\mathrm{q_C} > \frac{3}{2}$ and $d_{\mathrm{C}} = \mathrm{C}q_{\mathrm{C}}$, then

$$(q_{\mathrm{C}}^2 - d_{\mathrm{C}}^2)\lambda_i^{*2} + 2q_{\mathrm{C}}\lambda_i^* + 1 - 1 = \frac{9 - 6\mathrm{C}^2\mathrm{q_C}}{\mathrm{C}^4\mathrm{q_C^2}(1-\mathrm{C}^2)} > 0. \quad (20)$$

It is straightforward that $h^{''}(\lambda_i^*) > 0$. Consequently, λ_i^* is a minimum point. Then, replacing λ_i in (15) with λ_i^* gives (12) in Theorem 1 can be derived. □

Although a platoon achieves the consensus based on Theorem 1, it does not guarantee the absence of increasing spacing. Theorem 2 below provides the conditions according to string stability. Therefore, using the peak magnitude of spacing error $\tilde{p}_i(t)$, we define the string stability as below:

Definition 1. *[36] For a system of a platoon containing n vehicles, it is string stable if and only if*

$$\| \frac{\tilde{P}_i(j\omega)}{\tilde{P}_{i-1}(j\omega)} \|_\infty < 1, \forall \omega > 0, \quad (21)$$

where $\tilde{P}_i(s)$ is the Laplace transform of $\tilde{p}_i(t)$.

Theorem 2. *For a mixed platoon that contains two kinds of vehicles, i.e. C vehicles and HDVWs, the string stability based on different cases is guaranteed if the following conditions are met.*

Case 1: If the front vehicle of the C vehicle is a leader, and the rear vehicle is a C vehicle, the parameters of the C vehicle satisfy the following condition:

$$w_1^3 d_{\mathrm{C}}^2 + (q_{\mathrm{C}}^2 + 3d_{\mathrm{C}}^4 - 4d_{\mathrm{C}}^2 q_{\mathrm{C}})w_1^2 + (6q_{\mathrm{C}}^2 d_{\mathrm{C}}^2 - 4q_{\mathrm{C}}^3)w_1 + 3q_{\mathrm{C}}^4 > 0, \quad (22)$$

where $w_1 = \frac{\sqrt{4q_{\mathrm{C}}^3 d_{\mathrm{C}}^2 + q_{\mathrm{C}}^4} - q_{\mathrm{C}}^2}{d_{\mathrm{C}}^2}$.

Case 2: If the front vehicle of the C vehicle is a C vehicle, and the rear vehicle is a C vehicle, the parameters of the C vehicle satisfy the following condition:

$$w_2^3 d_{\mathrm{C}}^2 + (q_{\mathrm{C}}^2 + 5d_{\mathrm{C}}^4 - 6d_{\mathrm{C}}^2 q_{\mathrm{C}})w_2^2 + (10q_{\mathrm{C}}^2 d_{\mathrm{C}}^2 - 6q_{\mathrm{C}}^3)w_2 + 5q_{\mathrm{C}}^4 > 0, \quad (23)$$

where $w_2 = \frac{\sqrt{6q_{\mathrm{C}}^3 d_{\mathrm{C}}^2 + q_{\mathrm{C}}^4} - q_{\mathrm{C}}^2}{d_{\mathrm{C}}^2}$.

Case 3: If the front vehicle of the C vehicle is an HDVW, and the rear vehicle is a C vehicle, the parameters of the C vehicle satisfy the following condition:

$$w_2^3 d_{\mathrm{C}}^2 + (q_{\mathrm{C}}^2 - 6d_{\mathrm{C}}^2 q_{\mathrm{C}})w_2^2 - (12q_{\mathrm{C}}^2 d_{\mathrm{C}}^2 + 6q_{\mathrm{C}}^3)w_2 > 0. \quad (24)$$

Case 4: If the front vehicle of the C vehicle is a C vehicle, and the rear vehicle is an HDVW, the parameters of the C vehicle satisfy the following condition:

$$\begin{aligned} &4w_2^3 d_{\mathrm{C}}^2 + (4q_{\mathrm{C}}^2 + 27d_{\mathrm{C}}^4 - 24d_{\mathrm{C}}^2 q_{\mathrm{C}})w_2^2 \\ &+ (54q_{\mathrm{C}}^2 d_{\mathrm{C}}^2 - 24q_{\mathrm{C}}^3)w_2 + 27q_{\mathrm{C}}^4 > 0. \end{aligned} \quad (25)$$

Case 5: If the front vehicle of the last vehicle (the C vehicle) is a C vehicle, the parameters of the last vehicle satisfy the following condition:.

$$w_1^3 d_{\mathrm{C}}^2 + (q_{\mathrm{C}}^2 + 3d_{\mathrm{C}}^4 - 4d_{\mathrm{C}}^2 q_{\mathrm{C}})w_1^2 + (6q_{\mathrm{C}}^2 d_{\mathrm{C}}^2 - 4q_{\mathrm{C}}^3)w_1 + 3q_{\mathrm{C}}^4 > 0. \quad (26)$$

Case 6: If the front vehicle of the C vehicle is an HDVW, and the rear vehicle is an HDVW, the parameters of the C vehicle satisfy the following condition:

$$\begin{aligned} &4w_2^3 d_{\mathrm{C}}^2 + (4q_{\mathrm{C}}^2 + 9d_{\mathrm{C}}^4 - 24d_{\mathrm{C}}^2 q_{\mathrm{C}})w_2^2 \\ &+ (22q_{\mathrm{C}}^2 d_{\mathrm{C}}^2 - 24q_{\mathrm{C}}^3)w_2 + 9q_{\mathrm{C}}^4 > 0. \end{aligned} \quad (27)$$

Proof: See the Appendix B for the details. □

TABLE II
THE INITIAL STATES OF THE TWO PLATOONS

ith	types of the vehicle	$\tilde{p}_i(0)$ (km)	$\tilde{v}_i(0)$ (km/h)
1th	C	-0.01	2.49
2th	C	-0.03	1.00
3th	C	-0.04	0.82
4th	C	-0.03	-2.91
5th	C	-0.06	0.06
6th	HDVW		
7th	C	-0.07	-0.26
8th	C	-0.10	2.86
9th	C	-0.17	0.97
10th	C	-0.27	-1.14
11th	C	-0.30	0.93
12th	C	-0.33	-1.45
13th	C	-0.35	5.32
14th	C	-0.40	-1.50
15th	C	-0.45	3.79
16th	HDVW		
17th	C	-0.40	4.50
18th	C	-0.50	-2.85
19th	C	-0.60	-1.58
20th	C	-0.70	-2.69

V. Case study simulations

To verify the derived conclusions, in this section, the verification for the robustness of the platoon controllers is carried out under different scenarios. In addition, the robustness of the platoon controllers is analyzed by changing the penetration rate of the HDVWs. The simulations are performed via the Matlab (R2019a).

In this subsection, we first verify Theorem 1 with 2 predefined platoon scenarios. Then, the robustness of the controller is investigated by changing the penetration rate of the HDVWs. The safe distance between vehicle and in front of it should be maintained to reduce the risk of collision, and the vehicle spacing is constrained as follows:

$$p_i(t) - p_{i-1}(t) > \alpha_{\text{safe}}, \tag{28}$$

where $p_i(t)$ is the position of the ith vehicle at time t, and α_{safe} is the minimum safety distance.

A. Verification

In this subsection, we show the validation of effectiveness of Theorem 1 based on the two contrasting scenarios by defining different styles of communication in platoons. The initial states (speed errors, position errors and order) of the two platoons are the same, as shown in Table II, but the references are different. In Scenario 1, the coefficients satisfy the conditions in Theorem 1 and Theorem 2, while the coefficients in Scenario 2 do not satisfy these conditions. Specifically, the values of these coefficients are listed in Table III. In addition, the ideal distance between two adjacent vehicles is 40m.

Note that the position errors Δp is the position error between the ith and $(i-1)$th vehicles. It can be seen from Fig. 1 that the position errors between the ith and $(i-1)$th vehicles converge to 40m, in which the final values of 80m are because the front vehicles of the 7th and the 17th are the HDVWs, and the velocity errors and position errors of all the vehicles tend to 0 in the first scenario. For the second scenario as shown in

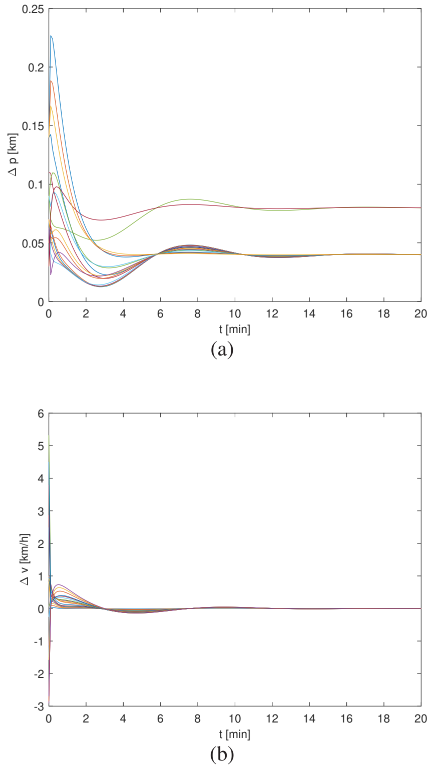


Fig. 1. Performances of the platoon satisfying Theorem 1 and Theorem 2:(a) the vehicle position errors varying with time; the vehicle velocity errors varying with time.

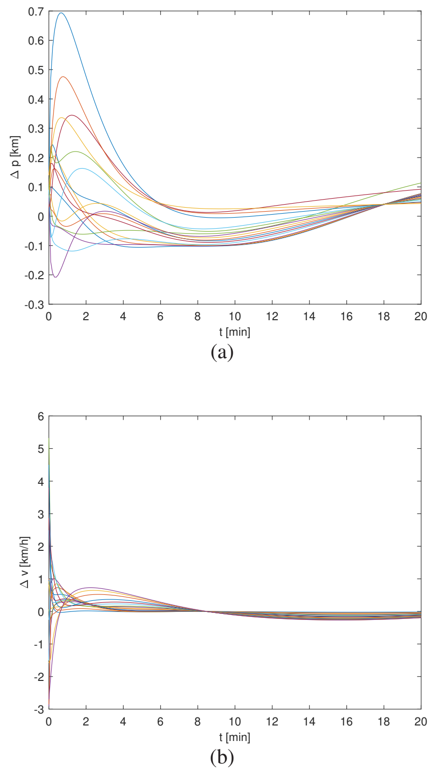
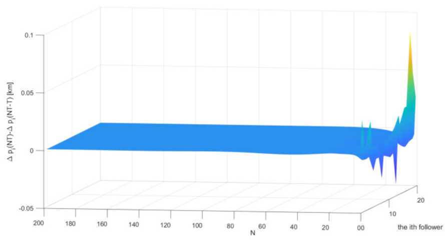


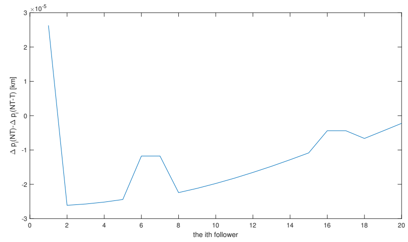
Fig. 2. Performances of the platoon not satisfying Theorem 1 and Theorem 2: (a) The vehicle position errors varying with time; (b) The vehicle velocity errors varying with time

TABLE III
LIST OF SYSTEM MODEL AND CONTROLLER PARAMETERS IN THE MODEL OF PLATOON FORMATION CONSENSUS USED FOR THE CASE STUDY

Symbol	Scenario 1	Scenario 2
d_{HDVW}	0.002	0.002
d_{C}	70.03	10
q_{C}	70	5

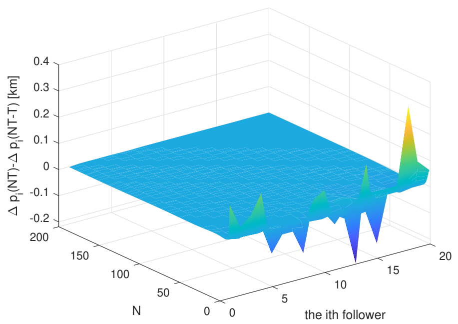


(a)

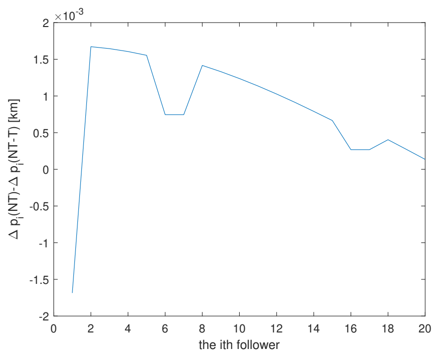


(b)

Fig. 3. Positions errors of the platoon satisfying Theorem 1 and Theorem 2:(a) the trend of the vehicle position errors varying with sampling time; (b) the trend of the follower vehicle velocity errors for the last sampling moment.

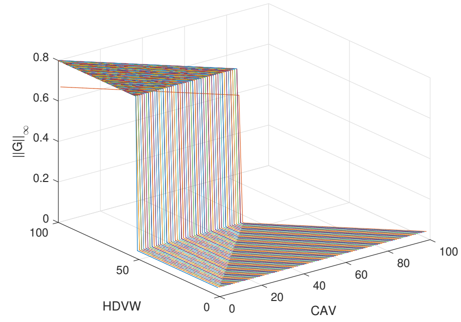


(a)

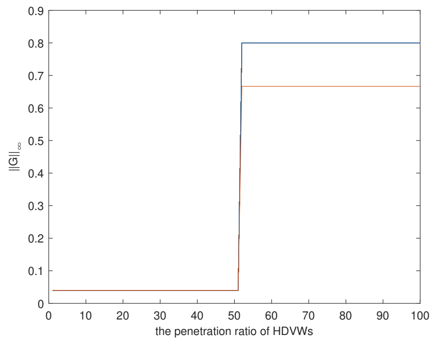


(b)

Fig. 4. Positions errors of the platoon not satisfying Theorem 1 and Theorem 2:(a) the trend of the vehicle position errors varying with sampling time; (b) the trend of the follower vehicle velocity errors for the last sampling moment.



(a)



(b)

Fig. 5. $\mathcal{H}_\infty$ norm of all vehicles under different penetration rates of HDVWs for platoon formation consensus: (a) $\mathcal{H}_\infty$ norm of all vehicles under different penetration rate of HDVWs (general layout); (b) $\mathcal{H}_\infty$ norm of all vehicles under different penetration rate of HDVWs with fixed proportion of P_{CAV} and P_{HDV}.

Fig. 2, although the velocity errors seem to converge to 0, the oscillations are more obvious than in Fig. 1. At the same time, the values of displacement errors in Fig. 2 are negative, which means that there is a collision in this platoon. The results verify that the given platoon system becomes unstable when Theorem 1 is not satisfied.

Fig. 3 and 4 display the trends of the position errors between the Nth and $(N-1)$th sampling moments for the followers satisfying or not satisfying Theorems 1 and 2 (N values from 1 to 200). In general, the performance of the Fig.3 is better than that of the Fig.4. Because the convergence speed of the platoon satisfying Theorems 1 and 2 in Fig.3(a) is better than that of the platoon not satisfying Theorems 1 and 2 in Fig.4(a), and the values of the position errors (except the 1st follower) on the curves are less than 0, i.e., the position errors will decrease along the vehicle platoon in Fig. 3(b).

B. $\mathcal{H}_\infty$ robustness under different penetration rates of the HDVWs

An dependent test was performed to study the impact of the penetration rate of the HDVWs on the robustness of platoon formation consensus.

The setup of this test is similar to the case of platoon velocity consensus. The total number of vehicles in the platoon is 100. The reference vehicle is located in front of the vehicles in the platoon, i.e. the leader. Then, for different penetration rate values of HDVWs and the CAVs, the $\mathcal{H}_\infty$ norm is calculated as shown in Fig. 5.

According to the simulation results, when the penetration rate of the HDVWs increases from 0% to 50%, the $\mathcal{H}_\infty$ norm of the system is low and flat, which means the controller is very robust to the uncertainties (i.e. the HDVWs). But, there is a turning point when the reactivation rate of the HDVWs increases to 50%, where the $\mathcal{H}_\infty$ norm values achieve a peak, i.e. the robustness of the controller dramatically drops. For a fixed penetration rate of the HDVWs, the higher the penetration rate of the CAVs is, the more robust the platoon controller is with respect to the uncertainty. Compared to the results of the platoon velocity consensus, the increase of the penetration rate of the CAVs contributes more to the system robustness.

VI. Conclusions

There is a transition stage, i.e. with mixed traffic flows of CAVs, HDVs, and HDVWs, before driverless technology will be fully deployed. This paper presents a method to quantify the robustness of the platoon control laws by an $\mathcal{H}_\infty$ norm against the uncertainties caused by HDVWs of the mixed platoons, based on graph theory. At the same time, the conditions for judging the stability of the control laws under uncertainties are proved for the formation consensus:

- First, we consider HDVWs as disturbances, and one state-space models is established to describe the dynamics of the velocities and the positions of vehicles.
- Second, by analyzing the upper bounds of the $\mathcal{H}_\infty$ norm for the platoon system, we present conditions to achieve velocity consensus and formation consensus. The conditions are proved mathematically, and the simulations demonstrate that the platoons can achieve formation consensus under the proposed controllers.
- Third, a new way for string stability analysis is proposed to effectively guarantee the security of the platoons.
- Finally, the robustness of the control law is analyzed by assessing variation of the $\mathcal{H}_\infty$ norm with the variation of the penetration rate of HDVWs.

In the future, we will study the consensus of the heterogeneous vehicular platoons consistency with more accurate team models. In addition, we will focus on coordination between platoons control and signal control to relieve traffic jams effectively.

Appendix

A. Proof of the Transfer Equations (6)

First, we obtain the following state space expression from (5):

$$\begin{aligned}\ddot{\tilde{\mathbf{p}}} &= \begin{bmatrix} B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{p}} \\ \dot{\tilde{\mathbf{p}}} \end{bmatrix} + \begin{bmatrix} \bar{I}_j & \bar{I}_j \end{bmatrix} Q \begin{bmatrix} \tilde{\mathbf{e}} \\ \dot{\tilde{\mathbf{e}}} \end{bmatrix} \\ &= -B_{21}\tilde{\mathbf{p}} - B_{22}\dot{\tilde{\mathbf{p}}} + q_{\text{HDVW}}\bar{I}_j\tilde{\mathbf{e}} + d_{\text{HDVW}}\bar{I}_j\dot{\tilde{\mathbf{e}}},\end{aligned} \tag{29}$$

where $Q = \begin{bmatrix} q_{\text{HDVW}} & \\ & d_{\text{HDVW}} \end{bmatrix}$.

As output signals of interest, we consider position errors of CAVs, HDVs as output. The input is the information of HDVWs. Based on the linear time-invariant control system as (5) and the theory of Laplace transform, (29) is rewritten with all initial condition assumed to be zero as:

$$\begin{aligned} s^2\tilde{\mathbf{P}}(s) &= -B_{21}\tilde{\mathbf{P}}(s) - B_{22}s\tilde{\mathbf{P}}(s) \\ &+ d_{\mathbf{HDVW}}\bar{I}_j s\tilde{\mathbf{E}}(s) + q_{\mathbf{HDVW}}\bar{I}_j\tilde{\mathbf{E}}(s), \end{aligned} \tag{30}$$

where $\tilde{\mathbf{P}}(s) = \mathscr{L}(\tilde{\mathbf{p}}(s))$, $\tilde{\mathbf{E}}(s) = \mathscr{L}(\tilde{\mathbf{e}}(s))$.

$$(s^2 + B_{22}s + B_{21})\tilde{\mathbf{P}}(s) = \left(d_{\mathbf{HDVW}}\bar{I}_j s + q_{\mathbf{HDVW}}\bar{I}_j\right)\tilde{\mathbf{E}}(s) \tag{31}$$

Hence, the transfer function is deduced as (6).

B. Proof of the Theorem 2

According to information transmitting of different kinds of vehicles, six cases are discussed as follows.

Case 1: In this case, the acceleration expression of the ith vehicle is as follows:

$$\begin{aligned}\ddot{p}_i(t) &= q_{\text{C}}(p_0(t) - p_i(t) + \Delta_{i0}) + d_{\text{C}}(v_0(t) - v_i(t)) \\ &+ q_{\text{C}}(p_{i+1}(t) - p_i(t) + \Delta_{i(i+1)}) + d_{\text{C}}(v_{i+1}(t) - v_i(t)).\end{aligned} \tag{32}$$

By introducing the position errors, the acceleration of the ith vehicle can be rewritten as the following:

$$\ddot{p}_i(t) = d_{\text{C}}\dot{p}_{i+1}(t) - 2d_{\text{C}}\dot{p}_i(t) + q_{\text{C}}\dot{p}_{i+1}(t) - 2q_{\text{C}}\dot{p}_i(t). \tag{33}$$

The Laplace form of the above equation is shown below:

$$P_i(s) = \frac{d_{\text{C}}s + q_{\text{C}}}{s^2 + 2d_{\text{C}}s + 2q_{\text{C}}}P_{i+1}(s). \tag{34}$$

The above equation can be rewritten as follows:

$$\frac{P_{i+1}(s)}{P_i(s)} = \frac{1}{Z_1(s)}, \tag{35}$$

where

$$Z_1(s) = \frac{d_{\text{C}}s + q_{\text{C}}}{s^2 + 2d_{\text{C}}s + 2q_{\text{C}}}.$$

According to the definition, string stability is obtained if (36) is satisfied.

$$\| Z_1 \|_\infty > 1. \tag{36}$$

Therefore, string stability is obtained if (37) is satisfied.

$$\| s^2 + 2d_{\text{C}}s + 2q_{\text{C}} \|_\infty < 1. \tag{37}$$

Let $Y_1(s) = \frac{s^2}{d_{\text{C}}s + q_{\text{C}}} + 2$. The amplitude of $Y_1(s)$ satisfies:

$$\| Y_1 \| = \frac{\sqrt{(-\omega^2 q_{\text{C}} + 2q_{\text{C}}^2 + 2d_{\text{C}}^2\omega^2)^2 + \omega^6 d_{\text{C}}^2}}{q_{\text{C}}^2 + d_{\text{C}}^2\omega^2}. \tag{38}$$

Consequently, $\omega_1^2 = \frac{\sqrt{4q_{\text{C}}^3 d_{\text{C}}^2 + q_{\text{C}}^4} - q_{\text{C}}^2}{d_{\text{C}}^2}$ is a minimum point.

The minimum value for $\| Y_1(j\omega) \|$ is

$$\frac{\sqrt{-d_{\text{C}}^2 w_1^3 + Ew_1^2 + (8q_{\text{C}}^2 d_{\text{C}}^2 - 4q_{\text{C}}^3)w_1^+ 4q_{\text{C}}^4}}{q_{\text{C}}^2 + d_{\text{C}}^2 w_1},$$

where $E = q_{\rm C}^2 + 4d_{\rm C}^4 - 4d_{\rm C}^2 q_{\rm C}$. According to (36), (22) can be derived.

Case 2: The acceleration expression of the ith vehicle is as follows:

$$\begin{aligned}\ddot{p}_i(t) = q_{\rm C}(p_0(t) - p_i(t) + \Delta_{i0}) + d_{\rm C}(v_0(t) - v_i(t)) \\ +q_{\rm C}(p_{i+1}(t) - p_i(t) + \Delta_{i(i+1)}) + d_{\rm C}(v_{i+1}(t) - v_i(t)) \\ +q_{\rm C}(p_{i-1}(t) - p_i(t) + \Delta_{i(i-1)}) + d_{\rm C}(v_{i-1}(t) - v_i(t)).\end{aligned} \tag{39}$$

By introducing the position errors, the acceleration of the ith vehicle can be rewritten as the following:

$$\begin{aligned}\ddot{\tilde{p}}_i(t) = d_{\rm C}\dot{\tilde{p}}_{i+1}(t) - 3d_{\rm C}\dot{\tilde{p}}_i(t) + d_{\rm C}\dot{\tilde{p}}_{i-1}(t) \\ +q_{\rm C}\tilde{p}_{i+1}(t) - 3q_{\rm C}\tilde{p}_i(t) + q_{\rm C}\tilde{p}_{i-1}(t).\end{aligned} \tag{40}$$

The Laplace form of the above equation is shown below:

$$\begin{aligned}P_i(s) = \frac{d_{\rm C}s + q_{\rm C}}{s^2 + 32d_{\rm C}s + 3q_{\rm C}} P_{i+1}(s) \\ + \frac{d_{\rm C}s + q_{\rm C}}{s^2 + 32d_{\rm C}s + 3q_{\rm C}} P_{i-1}(s).\end{aligned} \tag{41}$$

The above equation can be rewritten as follows:

$$\frac{P_i(s)}{P_{i-1}(s)} = \frac{Z_2(s)}{1 - Z_2(s)\dfrac{P_{i+1}(s)}{P_i(s)}}, \tag{42}$$

where

$$Z_2(s) = \frac{d_{\rm C}s + q_{\rm C}}{s^2 + 3d_{\rm C}s + 3q_{\rm C}}.$$

By assuming that $\parallel \frac{P_{i+1}(s)}{P_i(s)} \parallel < 1$ is satisfied, the string stability is achieved if $\parallel Z_2(s) \parallel_\infty < 0.5$ is met.If $\parallel Z_2(s) \parallel_\infty < 0.5$ is satisfied, then

$$\parallel \frac{s^2}{d_{\rm C}s + q_{\rm C}} + 3 \parallel_\infty > 2. \tag{43}$$

Let $Y_2(s) = \frac{s^2}{d_{\rm C}s + q_{\rm C}} + 3$. The amplitude of $Y_2(s)$ satisfies:

$$\parallel Y_2 \parallel = \frac{\sqrt{\omega^6 d_{\rm C}^2 + F\omega^4 + (18q_{\rm C}^2 d_{\rm C}^2 - 6q_{\rm C}^3)\omega^2 + q_{\rm C}^4}}{q_{\rm C}^2 + d_{\rm C}^2\omega^2}, \tag{44}$$

where $F = q_{\rm C}^2 + 9d_{\rm C}^4 - 6d_{\rm C}^2 q_{\rm C}$.

Consequently, The minimum value for $\parallel Y_2(j\omega) \parallel$ is

$$\frac{\sqrt{-w_2^3 d_{\rm C}^2 + Ew_2^2 + (8q_{\rm C}^2 d_{\rm C}^2 - 4q_{\rm C}^3)w_2 + 4q_{\rm C}^4}}{q_{\rm C}^2 + d_{\rm C}^2 w_2},$$

where $w_2 = \omega_2^2$. According to (43), (23) can be derived.

The proofs of cases 3, 4, 5 and 6 are similar to the above cases.

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