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Prescribed-Performance Tracking for High-Power Nonlinear Dynamics with Time-Varying Unknown Control Coefficients

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Abstract

Prescribed-performance control (PPC) for high-power dynamics with time-varying unknown control coefficients requires to address two open problems: a) given a Nussbaum function, which properties hold for the power of the Nussbaum function? b) to avoid high gains, how to design a switching gain that increases only when the tracking error is close to violate the performance bounds? To address the first problem, we show with a counterexample and a positive example that only some Nussbaum functions are suited to handle time-varying unknown control coefficients for high-power dynamics. To address the second problem, we propose a new switching conditional inequality.

Key words: Nussbaum function, High-power nonlinear dynamics, Adaptive Control, Switching conditional inequality, PPC

1 Introduction

Over the last decade, high-power nonlinear systems have been attracting great attention due to two reasons: first, high-power nonlinear systems generalize strict-feedback and pure-feedback systems by including more general odd-integer powers [1, 2, 3] in the dynamics; second, high-power nonlinear systems have been used to describe classes of practical systems such as dynamical boiler-turbine units [4], hydraulic dynamics [5], aircraft wing dynamics [6], or mechanical systems with cubic force-deformation relations [1, 2, 3]. The main technique for control of high-power nonlinear systems is the so-called adding-one-power-integrator technique, successfully used in several stabilization [1, 7] and tracking problems [2, 3]. However, handling unknown signs of constant or time-varying control coefficients [8, 9, 10, 11, 12, 13, 14, 15, 16], and guaranteeing transient and steady-state specifications [17, 18, 19] still pose open problems for high-power nonlinear systems, as explained hereafter.

The term "sign of the control coefficient" (also called "control direction" in some literature), refers to the sign of the control gain function. A control law in the presence of this uncertainty may apply its control action with an incorrect sign and destabilize the system [20, 21]. These signs have been assumed to be known until the celebrated method of Nussbaum [22], which proved stability with unknown signs using a special function (later called Nussbaum function) alternating its effects in both directions of the sign. Although alternative methods have been proposed to tackle unknown control coefficients, such as logic-based switching [23], nonlinear proportional-integral control [24], and extremum seeking [25], the Nussbaum function method is probably the most studied one. A fundamental tool to prove closed-loop stability with the Nussbaum function is the so-called conditional inequality, which consists in guaranteeing the boundedness of a Lyapunov-like function when its derivative along the system trajectories is upper bounded by an appropriate expression depending on the Nussbaum function. As the control coefficients can be constant or time-varying functions, three representative conditional inequalities have been proposed so far [20, 26, 27] to handle these cases. The first conditional inequality was formulated in [26] to handle unknown signs of constant control coefficients. The second conditional inequality in [27] (see also discussions in [28]) is given in integral form to handle unknown signs of time-varying control coefficients. Recently, [20] distinguished between type A and type B

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Nussbaum functions, where the former can handle constant control coefficients, but only the latter can handle time-varying control coefficients. Unfortunately, the capability to handle time-varying control coefficients was shown in [20] only for strict-feedback and pure-feedback systems. At the same time, it can be verified that combining the adding-one-power-integrator technique with Nussbaum functions [17] requires to take the derivative of the virtual control laws, which gives rise to negative fractional terms not well defined when the error crosses zero. Therefore, handling high-power nonlinear systems via the Nussbaum method is an open question due to the presence of positive odd-integer power in their dynamics.

With respect to guaranteeing transient and steady-state specifications (e.g. convergence rate, overshoot, or steady-state error), the prescribed-performance control (PPC) technique first [29] and low-complexity PPC later [30] have been successfully applied to strict-feedback [31, 32] and pure-feedback systems [30] with known signs of the control coefficients. To handle unknown signs, a low-complexity control scheme was recently developed in [33] for strict-feedback dynamics. Although the combination of the Nussbaum function method and PPC appears promising, one major challenge of this direction is to avoid high-gain issues, due to the presence of high powers. With these problems in mind, realizing Nussbaum PPC for high-power nonlinear dynamics with unknown signs of time-varying control coefficients requires to answer two open questions: *(i) is the positive odd-integer power of a type B Nussbaum function still a type B Nussbaum function? (ii) is it possible to design a different conditional inequality that may allow the Nussbaum gain to stop increasing over some time intervals?*

This paper answers these questions as follows:

- A counterexample and a positive example are given to show that the positive odd-integer-power of a type B Nussbaum function may not be a type B Nussbaum function. Only some particular type B Nussbaum functions keep their property even when elevated to a positive odd-integer power. These latter functions can be used for handling time-varying unknown control coefficients in high-power systems.

- A new switching conditional inequality is proposed. This inequality encompasses existing ones as special cases: instead of always increasing the Nussbaum gain, its design is based on increasing the Nussbaum gain only when the tracking error is close to violate the performance bounds.

2 Problem Formulation

This paper considers the high-power nonlinear systems:

$$\begin{cases} \dot{\chi}_i(t) = \phi_i(t, \overline{\boldsymbol{\chi}}_i) + \ell_i(t, \overline{\boldsymbol{\chi}}_i)\chi_{i+1}^{r_i}(t), \ i = 1, \ldots, n-1, \\ \dot{\chi}_n(t) = \phi_n(t, \overline{\boldsymbol{\chi}}_n) + \ell_n(t, \overline{\boldsymbol{\chi}}_n)u^{r_n}(t), \\ y(t) = \chi_1(t), \end{cases} \tag{1}$$

where $\overline{\boldsymbol{\chi}}_i = [\chi_1, \ldots, \chi_i]^T \in \mathbb{R}^i$, r_i, $i = 1, \ldots, n$, are known positive-odd integers, and $u \in \mathbb{R}$ is the control input. The unknown continuous nonlinear functions $\phi_i(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^i \to \mathbb{R}$ (referred to as drift coefficients) and $\ell_i(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^i \to \mathbb{R}$, $i = 1, \ldots, n$, (referred to as control coefficients) satisfy the following assumption.

Assumption 1 *[31] There exist unknown, continuous, and positive functions $\overline{\phi}_i(\cdot) : \mathbb{R}^i \to \mathbb{R}^+$, $\underline{\ell}_i(\cdot)$, and $\overline{\ell}_i(\cdot) : \mathbb{R}^i \to \mathbb{R}^+$, $i = 1, \ldots, n$, such that for all t*

$$|\phi_i(t, \overline{\boldsymbol{\chi}}_i)| \leq \overline{\phi}_i(\overline{\boldsymbol{\chi}}_i), \ \ \underline{\ell}_i(\overline{\boldsymbol{\chi}}_i) \leq |\ell_i(t, \overline{\boldsymbol{\chi}}_i)| \leq \overline{\ell}_i(\overline{\boldsymbol{\chi}}_i). \tag{2}$$

In line with standard literature on Nussbaum-based control [34, 35, 36], Assumption 1 allows the control coefficients $\ell_i(\cdot, \cdot)$ to be unknown but fixed, thus guaranteeing controllability of dynamics (1). Nussbaum-based definitions follow.

Definition 1 *[20, Definition 3.1], [22] A continuous function $\mathscr{N}(\cdot) : [0, +\infty) \to (-\infty, +\infty)$ is called a type A Nussbaum function if it satisfies*

$$\lim_{y \to +\infty} \sup \frac{\int_0^y \mathscr{N}(s) ds}{y} = +\infty, \lim_{y \to +\infty} \inf \frac{\int_0^y \mathscr{N}(s) ds}{y} = -\infty.$$

Definition 2 *[20, Definition 4.3] A continuous function $\mathscr{N}(\cdot) : [0, +\infty) \to (-\infty, +\infty)$ is called a type B Nussbaum function if it satisfies*

$$\lim_{y \to +\infty} \frac{\int_0^y \mathscr{N}_+(s) ds}{y} = +\infty, \lim_{y \to +\infty} \sup \frac{\int_0^y \mathscr{N}_-(s) ds}{\int_0^y \mathscr{N}_+(s) ds} = +\infty,$$

$$\lim_{y \to +\infty} \frac{\int_0^y \mathscr{N}_-(s) ds}{y} = +\infty, \lim_{y \to +\infty} \sup \frac{\int_0^y \mathscr{N}_+(s) ds}{\int_0^y \mathscr{N}_-(s) ds} = +\infty,$$

where $\mathscr{N}_+(s) = \max\{0, \mathscr{N}(s)\}$ and $\mathscr{N}_-(s) = \max\{0, -\mathscr{N}(s)\}$ are the positive and negative truncated functions of $\mathscr{N}(s)$.

Remark 1 *Note that type B Nussbaum functions are a special class of type A Nussbaum functions [20]. It was shown in [20] that type A Nussbaum functions can handle unknown signs of constant control coefficients, but may fail to handle unknown signs of time-varying control coefficients. Accordingly, type B Nussbaum functions were proposed to tackle the time-varying scenarios.*

The main problem studied in this paper is stated below.

Prescribed-performance control (PPC) problem: Consider a bounded reference signal $y_r(t)$ with bounded derivative and a performance function $\rho_1(t) = (\rho_{1,0} - \rho_{1,\infty})\exp(-\kappa_1 t) + \rho_{1,\infty}$ for positive constants $\rho_{1,0} > \rho_{1,\infty}$ and κ_1. The PPC problem aims to design a controller for the system (1) such that the closed-loop system satisfies the following two properties:

- **(P1)** The output tracking error $e_1(t) = y(t) - y_r(t)$ evolves in the prescribed set $\Omega = \{e_1(t) \in \mathbb{R} \mid |e_1(t)| < \rho_1(t)\}$ for $t \geq 0$; and
- **(P2)** The closed-loop signals are bounded on the entire time domain $[0, +\infty)$.

The PPC problem has been well formulated in literature, e.g., [29, 30]. However, this problem remains unsolved for the class of dynamics (1) and even the stability analysis recently proposed in [20] does not apply. Solving this problem requires to address two open issues: given a Nussbaum function, which properties hold for the power of the Nussbaum function? To avoid high gains, how to design a switching gain that increases only when the tracking error is close to violate the performance bounds? These two problems are addressed by the technical results in the next section.

3 Technical Results

The high-power terms in (1) require that the positive odd-integer power of a Nussbaum function, denoted by $\mathscr{N}^r(s)$, is still a Nussbaum function. However, we show that the positive odd-integer power of a type B Nussbaum function may not always result in a type B Nussbaum function. A counterexample and a positive example are given in the following two propositions, with the proofs given in Appendix.

Proposition 1 (Counterexample) *Consider the function*

$$\mathscr{N}(s) = \sum\nolimits_{\lambda \in \mathbb{N}^+} \mathscr{N}_\lambda(s + 2 - 2^\lambda), \tag{3}$$

where $\mathbb{N}^+$ is the set of positive integers and

$$\mathscr{N}_\lambda(s) = \begin{cases} 2^{(\lambda^3 + \frac{1}{3})\lambda} \sin(s\pi), & \text{if } s \in [0, 1) \\ -2^{\lambda^4} \sin\left(\dfrac{s-1}{2^\lambda - 1}\pi\right), & \text{if } s \in [1, 2^\lambda) \\ 0, \quad \text{otherwise.} \end{cases} \tag{4}$$

Then, $\mathscr{N}(\cdot)$ is a type B Nussbaum function, but $\mathscr{N}^r(\cdot)$ with $r \geq 3$ a positive odd integer is not *a type B Nussbaum function.*

Proposition 2 (Positive example) *Consider the function*

$$\mathscr{N}(s) = \exp(\mu s^2)\cos\left(\frac{\pi s}{2}\right), \ \mu > 0. \tag{5}$$

Then, $\mathscr{N}^r(\cdot)$ is a type B Nussbaum function for any positive odd integer $r \geq 1$.

Remark 2 *The above propositions may eventually lead to a new class of Nussbaum functions which are those functions where $\mathscr{N}^r(\cdot)$ satisfies Definition 2 for any positive odd integer r. The function (5), popular in Nussbaum-based control, belongs to such class.*

The following lemma is instrumental to constructing a Nussbaum gain that increases only when the tracking error is close to violate the performance bounds.

Lemma 1 (Switching conditional inequality) *Let $\mathscr{N}(\cdot)$ be a type B Nussbaum function. Consider two continuous and piecewise differentiable functions $V(\cdot)$ and $s(\cdot)$ such that*

$$\dot{V}(t) \leq [\ell(t)\mathscr{N}(s(t)) + \beta]\dot{s}(t), \tag{6}$$

$$\dot{s}(t) \begin{cases} \geq 0, & \text{if } V(t) \geq \phi, \\ = 0, & \text{if } V(t) < \phi, \end{cases} \tag{7}$$

where ϕ and β are positive constants, $V(0) < \phi$, $s(0) = 0$, and $\ell(\cdot)$ is a time-varying unknown function satisfying $\ell(t) \in [l_1, l_2]$, $\forall t$ with either $0 > l_2 > l_1$ or $l_2 > l_1 > 0$. Then, $V(\cdot)$ and $s(\cdot)$ are bounded on the entire time domain $[0, +\infty)$.

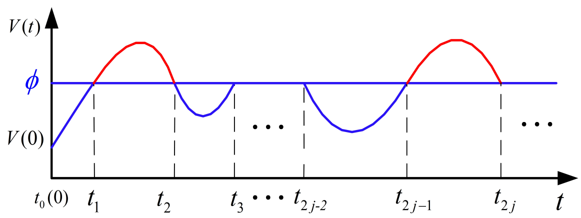

Figure 1. Illustration of the evolution of $V(\cdot)$.

Proof. For better comprehension, a sketch of the idea behind (7) is shown in Fig. 1. Let $0 = t_0 < t_1 \leq t_2 \leq t_3 \leq \ldots$ be the time sequence satisfying $V(t_j) = \phi$, $V(t) < \phi$, $\forall t \in (t_{2j-2}, t_{2j-1})$, and $V(t) \geq \phi$, $\forall t \in [t_{2j-1}, t_{2j}]$, for $j = 1, 2, \ldots$. According to the time sequence above, we consider the case of $t \in [t_{2m-1}, t_{2m}]$ for $m \in \mathbb{N}^+$. Integrating $\dot{V}(\cdot)$ over the time intervals $[t_0, t_1)$, $[t_1, t_2)$,

...., $[t_{2m-2}, t_{2m-1})$, $[t_{2m-1}, t]$ results in

$$
\begin{aligned}
V(t) \leq & \sum_{j=1}^{m-1} \int_{t_{2j-1}}^{t_{2j}} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt + \sum_{i=1}^{m} \int_{t_{2j-2}}^{t_{2j-1}} \dot{V}(t)dt \\
& + \phi + \int_{t_{2m-1}}^{t} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt \\
\leq & \phi + \sum_{j=1}^{m-1} \int_{t_{2j-1}}^{t_{2j}} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt \\
& + \int_{t_{2m-1}}^{t} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt,
\end{aligned}
$$

where the integral over $t \in [t_{2j-2}, t_{2j-1})$ has been removed by observing that $V(t_{2j-1}) = V(t_{2j-2}) = \phi$. Then, it follows that

$$
\begin{aligned}
V(t) \leq & \phi + \sum_{j=1}^{m-1} \int_{t_{2j-1}}^{t_{2j}} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt \\
& \underbrace{+ \sum_{j=1}^{m-1} \int_{t_{2j-2}}^{t_{2j-1}} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt}_{\Theta(s(t))} \\
& + \int_{t_{2m-1}}^{t} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt \\
\leq & \phi + \int_{0}^{t} \Big[\ell(t)\mathscr{N}(s(t)) + \beta\Big]\dot{s}(t)dt \\
\leq & \phi + \beta s(t) + \underbrace{l_2 \int_0^{s(t)} \mathscr{N}_{+}(\tau)d\tau - l_1 \int_0^{s(t)} \mathscr{N}_{-}(\tau)d\tau}_{\Xi(s(t))},
\end{aligned}
\tag{8}
$$

by noting the facts that $\Theta(s(t)) \equiv 0$ due to $\dot{s}(t) = 0$ for $t \in [t_{2j-2}, t_{2j-1}]$, $s(0) = 0$, and $\mathscr{N}(s) = \mathscr{N}_{+}(s) - \mathscr{N}_{-}(s)$. When $s(t) = 0$, $\forall t$, the boundedness of $s(t)$ and $V(t)$ can be trivially obtained according to (8).

When $s(t) \neq 0$, it is obtained from (8) that

$$
0 \leq \frac{V(t)}{s(t)} \leq \underbrace{\overbrace{\left[\frac{\Xi(s(t))}{s(t)}\right]}^{\Delta(s(t))} + \frac{\phi}{s(t)} + \beta}_{\Upsilon(s(t))}. \tag{9}
$$

In the following, we aim to prove boundedness of $s(\cdot)$ on $[0, +\infty)$ by contradiction. If $s(\cdot)$ is unbounded, one can calculate the limit behavior of $\Delta(s)$ in (9) as $s \to +\infty$, using the Definition 2. In particular, for the case of $0 > l_2 > l_1$,

$$
\begin{aligned}
& \lim_{s\to+\infty} \inf \Delta(s) \\
= & \lim_{s\to+\infty} \overbrace{\frac{1}{s}\int_0^s \mathscr{N}_{-}(\tau)d\tau}^{\to+\infty} \overbrace{\left[-l_1 + \underbrace{l_2 \sup\frac{\int_0^s \mathscr{N}_{+}(\tau)d\tau}{\int_0^s \mathscr{N}_{-}(\tau)d\tau}}_{\to-\infty}\right]}^{\to-\infty} \\
= & -\infty,
\end{aligned}
\tag{10}
$$

and similarly, for the case of $l_2 > l_1 > 0$,

$$
\begin{aligned}
& \lim_{s\to+\infty} \inf \Delta(s) \\
= & \lim_{s\to+\infty} \overbrace{\frac{1}{s}\int_0^s \mathscr{N}_{+}(\tau)d\tau}^{\to+\infty} \overbrace{\left[l_2 \underbrace{-l_1 \sup\frac{\int_0^s \mathscr{N}_{-}(\tau)d\tau}{\int_0^s \mathscr{N}_{+}(\tau)d\tau}}_{\to-\infty}\right]}^{\to-\infty} \\
= & -\infty.
\end{aligned}
\tag{11}
$$

Note that 'inf' in (10) and (11) becomes 'sup' due to $l_2 < 0$ and $l_1 > 0$, respectively. The relations above indicate that an unbounded s leads to a negative unbounded $\Delta(s)$. Independently of whether the unboundedness of $\Delta(s)$ occurs in finite time or at infinity (this depends on the behavior of $s(\cdot)$), the consequence would be that there exists a time $\bar{t} > 0$ such that

$$
\Upsilon\big(s(\bar{t})\big) \leq -\varepsilon
$$

for some positive ε, which contradicts (9). It concludes that $s(\cdot)$ is bounded over the entire time domain $[0, +\infty)$, so are $\Xi(s(\cdot))$ and hence $V(\cdot)$ from (8).

Finally, let us now consider the case of $t \in (t_{2m}, t_{2m+1})$. The boundedness of $s(\cdot)$ and $V(\cdot)$ is guaranteed by the above argument for $t = t_{2m}$ and the facts that $V(t) < \phi$ and $\dot{s}(t) = 0$ for $t \in (t_{2m}, t_{2m+1})$. ■

Remark 3 *Lemma 1 encompasses [20, Lemma 4.3] as special case when* $\dot{s}(t) = 0$ *in* (7) *is never active (e.g. when* ϕ *is sufficiently small). Also, while existing conditional inequalities [27, Lemma 2], [9, Lemma 1], and [11, Lemma 2] guarantee boundedness on a finite time interval* $[0, t_\delta)$ *with* $t_\delta < +\infty$ *(cf. discussion in [28, Remark 1]), the proposed Lemma 1 can ensure boundedness on the entire time domain* $[0, +\infty)$*. This is due to the properties of type B Nussbaum functions used in the proof by contradiction (cf.* (10)-(11)*).*

4 Nussbaum Gain Adaptive PPC Design

The development of this section starts with the performance functions $\rho_i(t) = (\rho_{i,0} - \rho_{i,\infty})\exp(-\kappa_i t) + \rho_{i,\infty}$ for positive constants $\rho_{i,0} > \rho_{i,\infty}$ and κ_i, $i = 1, \ldots, n$.

In line with [29, 30, 31, 32], the initial conditions $e_i(0)$ should satisfy the initial feasibility $|e_i(0)| < \rho_i(0)$, $i = 1, \ldots, n$, i.e. start inside the prescribed performance. Let $\alpha_1(t) = y_{\mathrm{r}}(t)$, $\alpha_{i+1}(t)$, $i = 1, \ldots, n$, be the virtual control laws to be designed, and $u(t) = \alpha_{n+1}(t)$ be the real control law.

Next, we introduce the virtual tracking error $e_i(t) = \chi_i(t) - \alpha_i(t)$ and the error transformation

$$\mathcal{T}_i(t) = \frac{\tan\left(\frac{\pi}{2}\frac{e_i(t)}{\rho_i(t)}\right)}{\cos^2\left(\frac{\pi}{2}\frac{e_i(t)}{\rho_i(t)}\right)}, \ i = 1, \ldots, n. \tag{12}$$

The virtual control functions are devised as follows,

$$\alpha_{i+1}(t) = \varrho_i \mathscr{N}\big(s_i(t)\big)\mathcal{T}_i(t), \ i = 1, \ldots, n, \tag{13}$$

where $\varrho_i > 0$ is a design parameter and $\mathscr{N}^r(\cdot)$ is a type B Nussbaum function for any positive odd integer $r \geq 1$. An adaptation law for $s_i(t)$ is constructed as

$$\dot{s}_i(t) = \begin{cases} \mathcal{T}_i^{r_i+1}(t), & \text{if } |e_i(t)| \geq \delta_i \rho_i(t) \\ 0, & \text{if } |e_i(t)| < \delta_i \rho_i(t) \end{cases} \tag{14}$$

for a constant $\delta_i \in (0, 1)$. Similarly to [31, eq. (8)], equation (14) increases only when tracking error is close to violate the performance bound: however, the stability analysis in [31] is for strict-feedback dynamics and cannot be used to prove stability here. The way to prove the stability of this mechanism relies on the proposed Lemma 1. Before moving on, we give a technical lemma that is similar to [31, Lemma 3] and the proof is thus omitted. Then, the main result is stated.

Lemma 2 *If $\overline{\boldsymbol{\chi}}_i(\cdot)$, $\dot{\alpha}_i(\cdot)$, $s_i(\cdot)$, $\mathcal{T}_i(\cdot)$, and $e_{i+1}(\cdot)$ are bounded on a time interval $[0, t_\delta)$ with t_δ a strictly positive time instant, then $\dot{\alpha}_{i+1}(\cdot)$ is bounded on $[0, t_\delta)$ for $i = 1, \ldots, n$.*

Theorem 1 *Under Assumption 1 and with $\mathscr{N}^r(\cdot)$ being a type B Nussbaum function for any positive odd integer $r \geq 1$, consider the closed-loop system composed of (1), the control laws (12)-(13), and the adaptation law (14). In particular, $\mathscr{N}^r(\cdot)$ is a type B Nussbaum function for any positive odd integer $r \geq 1$. If the initial conditions $e_i(0)$ satisfy $|e_i(0)| < \rho_i(0)$, $i = 1, \ldots, n$, then the PPC problem is solved in the sense of P1 and P2.*

Proof. (Time dependence of the functions e_i, α_i, $\dot{\alpha}_i$, and $\mathscr{N}(s_i)$ will be omitted whenever unambiguous). Taking the time derivative of e_i along (1), (12) and (13) yields

$$\begin{aligned} \dot{e}_i &= \dot{\chi}_i - \dot{\alpha}_i = \phi_i(t, \overline{\boldsymbol{\chi}}_i) + \ell_i(t, \overline{\boldsymbol{\chi}}_i)(e_{i+1} + \alpha_{i+1})^{r_i} - \dot{\alpha}_i \\ &= \phi_i(t, \overline{\boldsymbol{\chi}}_i) + \ell_i(t, \overline{\boldsymbol{\chi}}_i)\vartheta_i(e_{i+1}, \alpha_{i+1})e_{i+1}^{r_i} - \dot{\alpha}_i \\ &\quad + \ell_i(t, \overline{\boldsymbol{\chi}}_i)\gamma_i(e_{i+1}, \alpha_{i+1})\alpha_{i+1}^{r_i} \\ &= F_i(t) + \gamma_i(e_{i+1}, \alpha_{i+1})\ell_i(t, \overline{\boldsymbol{\chi}}_i)\varrho_i^{r_i}\mathscr{N}^{r_i}(s_i)\mathcal{T}_i^{r_i}(t), \\ \dot{e}_n &= F_n(t) + \ell_n(t, \overline{\boldsymbol{\chi}}_n)\varrho_n^{r_n}\mathscr{N}^{r_n}(s_n)\mathcal{T}_n^{r_n}(t), \end{aligned} \tag{15}$$

where the second equality used the separation lemma of [37, 38], $|\vartheta_i(e_{i+1}, \alpha_{i+1})| \leq \bar{\vartheta}_i$ with $\bar{\vartheta}_i$ a positive constant, $\gamma_i(e_{i+1}, \alpha_{i+1}) \in [1 - \bar{\epsilon}_i, 1 + \bar{\epsilon}_i]$ with an arbitrary constant $\bar{\epsilon}_i \in (0, 1)$, $F_i(t) = \phi_i(t, \overline{\boldsymbol{\chi}}_i) + \ell_i(t, \overline{\boldsymbol{\chi}}_i)\vartheta_i(e_{i+1}, \alpha_{i+1})e_{i+1}^{r_i} - \dot{\alpha}_i$, $i = 1, \ldots, n-1$, and $F_n(t) = \phi_n(t, \overline{\boldsymbol{\chi}}_n) - \dot{\alpha}_n$.

In what follows, we will prove that $|e_i(t)| < \rho_i(t)$, $i = 1, \ldots, n$, holds for $t \geq 0$ using a contradiction. Suppose there exists an error e_m such that

$$|e_m(t_m)| \geq \rho_m(t_m), \ \ \forall m \in \{1, \ldots, n\}. \tag{16}$$

Let $t_\delta = \min\{t_m\}$ be the time instant when (16) is violated for the first time. Then, due to the continuity of e_i and the fact that $|e_i(0)| < \rho_i(0)$, $i = 1, \ldots, n$, it follows that

$$|e_i(t)| < \rho_i(t), \ \ \forall t \in [0, t_\delta), \tag{17}$$

and that there exists an error e_δ satisfying

$$\lim\nolimits_{t \to t_\delta^-} |e_\delta(t)| = \lim\nolimits_{t \to t_\delta^-} |\rho_\delta(t)|, \ \ \delta \in \{1, \ldots, n\}, \tag{18}$$

where t_δ^- denotes the left limit of t_δ.
To seek a contradiction, the analysis given below is conducted on a finite time interval $[0, t_\delta)$.

Step 1: Consider the Lyapunov function candidate

$$V_1(t) = \frac{1}{2}\tan^2\left(\frac{\pi}{2}\frac{e_1(t)}{\rho_1(t)}\right), \ \ \forall t \in [0, t_\delta). \tag{19}$$

When $|e_1(t)| < \delta_1\rho_1(t)$, it immediately follows that

$$V_1(t) < \frac{1}{2}\tan^2\left(\frac{\pi\delta_1}{2}\right) \triangleq \overline{\psi}_1. \tag{20}$$

From (14), we further have

$$\dot{s}_1(t) = 0, \ \ \text{when } V_1(t) < \overline{\psi}_1. \tag{21}$$

When $|e_1(t)| \geq \delta_1\rho_1(t)$, $V_1(t) \geq \overline{\psi}_1$ holds. Taking the time derivative of $V_1(t)$ along (15) yields

$$\begin{aligned} \dot{V}_1(t) &= \frac{\pi}{2}\frac{\mathcal{T}_1(t)}{\rho_1^2(t)}\Big[\dot{e}_1(t)\rho_1(t) - e_1(t)\dot{\rho}_1(t)\Big] \\ &= \mathcal{T}_1(t)F_{1f}(t) + g_{1f}(t)\mathscr{N}^{r_1}(s_1)\mathcal{T}_1^{r_1+1}(t), \end{aligned} \tag{22}$$

where

$$F_{1f}(t) = \frac{\pi}{2}\left(\frac{F_1(t)}{\rho_1(t)} - \frac{e_1(t)\dot{\rho}_1(t)}{\rho_1^2(t)}\right),$$
$$g_{1f}(t) = \frac{\pi}{2}\frac{1}{\rho_1(t)}\gamma_1(e_2,\alpha_2)\ell_1(t,\chi_1)\varrho_1^{r_1}.$$

According to the boundedness of y_r and its derivative, $\alpha_1(\cdot)$ and $\dot{\alpha}_1(\cdot)$ are bounded on $[0,t_\delta)$, which, together with (17), yields the boundedness of $\chi_1(\cdot)$ on $[0,t_\delta)$. By Assumption 1, the boundedness of χ_1 and $\dot{\alpha}_1$ results in that of $F_1(\cdot)$ and hence $F_{1f}(\cdot)$ on $[0,t_\delta)$. Invoking the boundedness of $\gamma_1(e_2,\alpha_2)$, $\rho_1(\cdot)$, and $\ell_1(\cdot,\chi_1)$ leads to the boundedness of $g_{1f}(\cdot)$ on $[0,t_\delta)$. Then, it follows from the Extreme Value Theorem that there exist positive constants $\bar{F}_{1f}$, $\underline{g}_{1f}$, and $\bar{g}_{1f}$ such that

$$|F_{1f}(t)| \le \bar{F}_{1f},\ g_{1f}(t) \in [\underline{g}_{1f},\bar{g}_{1f}],\ 0 \notin [\underline{g}_{1f},\bar{g}_{1f}]. \quad (23)$$

Substituting $|e_1(t)| \ge \delta_1\rho_1(t)$ into (12) gives

$$\left|\mathcal{T}_1^{r_1}(t)\right| \ge \frac{\tan^{r_1}\left(\frac{\pi}{2}\delta_1\right)}{\cos^{2r_1}\left(\frac{\pi}{2}\delta_1\right)} \ge \tan^{r_1}\left(\frac{\pi}{2}\delta_1\right). \quad (24)$$

Synthesizing (22)-(24) results in

$$\begin{aligned}\dot{V}_1(t) &\le \frac{|F_{1f}(t)|}{|\mathcal{T}_1^{r_1}(t)|}\mathcal{T}_1^{r_1+1}(t) + g_{1f}(t)\mathscr{N}^{r_1}(s_1)\mathcal{T}_1^{r_1+1}(t)\\ &\le \left[\frac{\bar{F}_{1f}}{\tan^{r_1}\left(\frac{\pi}{2}\delta_1\right)} + g_{1f}(t)\mathscr{N}^{r_1}(s_1)\right]\dot{s}_1(t). \quad (25)\end{aligned}$$

It is noted from Proposition 5 that $\mathscr{N}^{r_1}(\cdot)$ is a type B Nussbaum function. So, we can apply Lemma 1 to prove that $V_1(\cdot)$ and $s_1(\cdot)$ are bounded on $[0,t_\delta)$. In view of (19), we can claim that there exists a constant $\bar{\sigma}_1 > 0$ such that $|e_1(t)| \le \rho_1(t) - \bar{\sigma}_1 < \rho_1(t), \forall t \in [0,t_\delta)$ (equivalently to the boundedness of $\mathcal{T}_1(\cdot)$ on $[0,t_\delta)$). This, together with (12) and the boundedness of $\mathscr{N}(s_1)$, gives the boundedness of $\alpha_2(\cdot)$ and $\overline{\boldsymbol{\chi}}_2(\cdot)$ on $[0,t_\delta)$ due to $\chi_i = e_i + \alpha_i$, $i = 1,2$. By Lemma 2, $\dot{\alpha}_2(\cdot)$ is bounded on $[0,t_\delta)$.

Step $i (i = 2,\ldots,n)$**:** Boundedness of $\overline{\boldsymbol{\chi}}_i(\cdot)$ and $\dot{\alpha}_i(\cdot)$ on $[0,t_\delta)$ was obtained from step $i-1$. Consider the Lyapunov function candidate

$$V_i(t) = \frac{1}{2}\tan^2\left(\frac{\pi}{2}\frac{e_i(t)}{\rho_i(t)}\right), \quad \forall t \in [0,t_\delta). \quad (26)$$

When $|e_i(t)| < \delta_i\rho_i(t)$, it follows that

$$V_i(t) < \frac{1}{2}\tan^2\left(\frac{\pi\delta_i}{2}\right) \triangleq \overline{\psi}_i. \quad (27)$$

From (14) one has

$$\dot{s}_i(t) = 0, \quad \text{when } V_i(t) < \overline{\psi}_i. \quad (28)$$

When $|e_i(t)| \ge \delta_i\rho_i(t)$, it holds that $V_i(t) \ge \overline{\psi}_i$. Taking the time derivative of $V_i(t)$ along (14) gives

$$\begin{aligned}\dot{V}_i(t) &= \frac{\pi}{2}\frac{\mathcal{T}_i(t)}{\rho_i^2(t)}\left[\dot{e}_i(t)\rho_i(t) - e_i(t)\dot{\rho}_i(t)\right]\\ &= \mathcal{T}_i(t)F_{if}(t) + g_{if}(t)\mathscr{N}^{r_i}(s_i)\mathcal{T}_i^{r_i+1}(t), \quad (29)\end{aligned}$$

where

$$F_{if}(t) = \frac{\pi}{2}\left(\frac{F_i(t)}{\rho_i(t)} - \frac{e_i(t)\dot{\rho}_i(t)}{\rho_i^2(t)}\right),$$
$$g_{if}(t) = \frac{\pi}{2}\frac{1}{\rho_i(t)}\gamma_i(e_{i+1},\alpha_{i+1})\ell_i(t,\overline{\boldsymbol{\chi}}_i)\varrho_i^{r_i}.$$

In light of Assumption 1 and the boundedness of $\overline{\boldsymbol{\chi}}_i(\cdot)$, $\dot{\alpha}_i(\cdot)$ and $e_{i+1}(\cdot)$ on $[0,t_\delta)$, $F_i(\cdot)$ is bounded on $[0,t_\delta)$, which further ensures the boundedness of $F_{if}(\cdot)$ on $[0,t_\delta)$. Recalling Assumption 1 and the boundedness of $\gamma_i(e_{i+1},\alpha_{i+1})$ leads to that of $g_{if}(\cdot)$ on $[0,t_\delta)$. Similar to Step 1, one can conclude there exist positive constants $\bar{F}_{if}$, $\underline{g}_{if}$, and $\bar{g}_{if}$ such that

$$|F_{if}(t)| \le \bar{F}_{if},\ g_{if}(t) \in [\underline{g}_{if},\bar{g}_{if}],\ 0 \notin [\underline{g}_{if},\bar{g}_{if}]. \quad (30)$$

Substituting $|e_i(t)| \ge \delta_i\rho_i(t)$ into (12) results in

$$\left|\mathcal{T}_i^{r_i}(t)\right| \ge \frac{\tan^{r_i}\left(\frac{\pi}{2}\delta_i\right)}{\cos^{2r_i}\left(\frac{\pi}{2}\delta_i\right)} \ge \tan^{r_i}\left(\frac{\pi}{2}\delta_i\right). \quad (31)$$

Summarizing (29)-(31) leads to

$$\begin{aligned}\dot{V}_i(t) &\le \frac{|F_{if}(t)|}{|\mathcal{T}_i^{r_i}(t)|}\mathcal{T}_i^{r_i+1}(t) + g_{if}(t)\mathscr{N}^{r_i}(s_i)\mathcal{T}_i^{r_i+1}(t)\\ &\le \left[\frac{\bar{F}_{if}}{\tan^{r_i}\left(\frac{\pi}{2}\delta_i\right)} + g_{if}(t)\mathscr{N}^{r_i}(s_i)\right]\dot{s}_i(t). \quad (32)\end{aligned}$$

Likewise, $\mathscr{N}^{r_i}(\cdot)$ is a type B Nussbaum function, so we apply Lemma 1 to prove that $V_i(\cdot)$ and $s_i(\cdot)$ are bounded on $[0,t_\delta)$. According to (26), there exists a constant $\bar{\sigma}_i > 0$ such that $|e_i(t)| \le \rho_i(t) - \bar{\sigma}_i < \rho_i(t)$ $\forall t \in [0,t_\delta)$, which, combined with (12) and the boundedness of $\mathscr{N}(s_i)$, yields the boundedness of $\alpha_{i+1}(\cdot)$ and $\overline{\boldsymbol{\chi}}_{i+1}(\cdot)$ on $[0,t_\delta)$ owing to $\chi_{i+1} = e_{i+1}+\alpha_{i+1}$. Therefore, $\dot{\alpha}_{i+1}(\cdot)$ is bounded on $[0,t_\delta)$ according to Lemma 2.

In summary, we have proved that $|e_i(t)| \le \rho_i(t) - \bar{\sigma}_i < \rho_i(t)$, $i = 1,\ldots,n$, for $t \in [0,t_\delta)$. However, this contradicts the assumption made in (18) and implies that t_δ should be extended to $+\infty$. As a result, $|e_i(t)| < \rho_i(t)$, $i = 1,\ldots,n$, holds for $t \in [0,+\infty)$. Given that Lemma 1 holds true on $[0, +\infty)$, the boundedness of closed-loop signals is guaranteed on $[0, +\infty)$. This completes the proof. ■

5 Simulation Verification

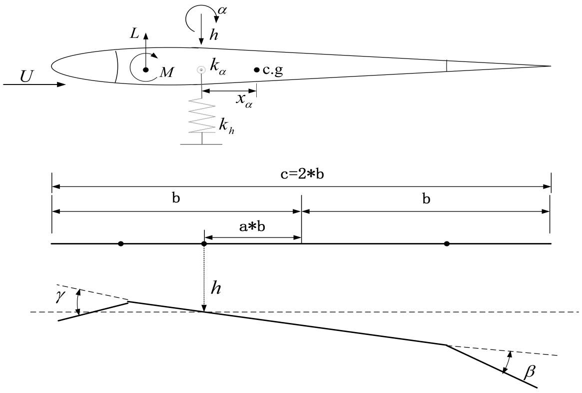

Figure 2. Wing section with leading-edge (LE) and trailing-edge (TE) control surfaces.

To validate the proposed method, a two-degree-of-freedom wing section with leading-edge (LE) and trailing-edge (TE) control surfaces as shown in Fig. 2 is considered. The dynamic of this aeroelastic system can be described by [6, 39]:

$$\begin{bmatrix} I_\alpha & m_{\mathrm{w}} x_\alpha b \\ m_{\mathrm{w}} x_\alpha b & m_t \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{h} \end{bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\alpha(\dot{\alpha}) \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{h} \end{bmatrix} + \begin{bmatrix} k_\alpha(\alpha) & 0 \\ 0 & k_h(h) \end{bmatrix} \begin{bmatrix} \alpha \\ h \end{bmatrix} = \begin{bmatrix} M \\ -L \end{bmatrix} \tag{33}$$

where α and h denote the pitch angle and the plunge displacement, respectively; I_α is the moment of inertia; $m_{\mathrm{w}} = m_{\mathrm{t}} + m_{\mathrm{l}}$ is the sum of wing section mass m_{t} and load section mass m_{l}; x_α is the distance between the center of mass and the elastic axis; b is the semi-chord of the wing; c_h is the plunge damping coefficient. The pitch damping $c_\alpha(\dot{\alpha})$, the pitch stiffness $k_\alpha(\alpha)$, and the plunge stiffness $k_h(h)$ are expressed as $c_\alpha(\dot{\alpha}) = \sum_{j=0}^{2} c_{\alpha j} \dot{\alpha}^j$, $k_\alpha(\alpha) = \sum_{j=0}^{2} k_{\alpha j} \alpha^j$, and $k_h(h) = \sum_{j=0}^{2} k_{hj} h^j$, where $c_{\alpha j}$, $k_{\alpha j}$, and k_{hj} are unknown non-zero constants, so they cannot be used in control design. In (33), M and L represent the aerodynamic moment and lift expressed by

$$M = \rho U^2 b^2 s_p \left\{ \bar{c}_{l_\alpha} \left(\alpha + \frac{\dot{h}}{U} + (0.5 - a)\, b \frac{\dot{\alpha}}{U} \right) + \bar{c}_{l_\beta} \beta + \bar{c}_{l_\gamma} \gamma \right\}$$
$$L = \rho U^2 b s_p \left\{ c_{l_\alpha} \left(\alpha + \frac{\dot{h}}{U} + (0.5 - a)\, b \frac{\dot{\alpha}}{U} \right) + c_{l_\beta} \beta + c_{l_\gamma} \gamma \right\}$$

where $\bar{c}_{l_\alpha} = \left(\frac{1}{2} + a\right) c_{l_\alpha} + 2c_{m_\alpha}$, $\bar{c}_{l_\beta} = \left(\frac{1}{2} + a\right) c_{l_\beta} + 2c_{m_\beta}$, $\bar{c}_{l_\gamma} = \left(\frac{1}{2} + a\right) c_{l_\gamma} + 2c_{m_\gamma}$, and ρ is the air density; U denotes the freestream velocity; c_{l_α}, c_{l_β} and c_{l_γ} are the lift derivatives; c_{m_α}, c_{m_β} and c_{m_γ} are the moment derivatives; s_p is the span; a is the nondimensional distance from midchord to the elastic axis; β and γ are the TE and LE control surface deflections, respectively. With the change of coordinates $\chi_1 = \alpha$, $\chi_2 = \dot{\alpha}$, $\chi_3 = h$, $\chi_4 = \dot{h}$, and $u = \beta + \gamma$, we can rewrite (33) as

$$\begin{cases} \dot{\chi}_1 = \chi_2, & \dot{\chi}_2 = \phi_2(\bar{\chi}_2) + \ell_2(\bar{\chi}_2)\chi_3^3, \\ \dot{\chi}_3 = \chi_4, & \dot{\chi}_4 = \phi_4(\bar{\chi}_4) + u, \end{cases} \tag{34}$$

where $\phi_2(\chi) = c_{\bar{\alpha}_1}\chi_1 + c_{\alpha_{11}}\chi_1^3 + c_{\dot{\alpha}_1}\chi_2 + c_{\dot{\alpha}_{11}}\chi_2^3 + c_{\dot{h}_1}\chi_2 + c_{\beta_1}\beta + c_{\gamma_1}\gamma$, $\phi_4(\chi) = c_{\alpha_2}\chi_1 + c_{\alpha_{21}}\chi_1^3 + c_{\dot{\alpha}_2}\chi_2 + c_{\dot{\alpha}_{21}}\chi_2^3 + c_{h_{21}}\chi_3^3 + c_{\dot{h}_2}\chi_4$, and $\ell_2(\bar{\chi}_2) = m_{\mathrm{w}} x_\alpha b k_{h2}$ with $c_{\bar{\alpha}_1} = c_2 m_{\mathrm{t}} c_{m\alpha} + c_1 m_{\mathrm{t}} x_\alpha b c_{l\alpha}$, $c_{\alpha_{11}} = -m_{\mathrm{t}} k_{\alpha_2}$, $c_{\dot{\alpha}_1} = c_2 m_{\mathrm{t}} c_{m\alpha} (0.5 - a) \frac{b}{U} - c_{\alpha 0} m_{\mathrm{t}} + c_1 m_{\mathrm{t}} x_\alpha b c_{l\alpha} (0.5 - a) \frac{b}{U}$, $c_{\dot{\alpha}_{11}} = -m_{\mathrm{t}} c_{\alpha 2}$, $c_{\dot{h}_1} = c_2 m_{\mathrm{t}} c_{m\alpha} \frac{1}{U} + c_1 m_{\mathrm{t}} x_\alpha b c_{l\alpha} \frac{1}{U} - c_h m_{\mathrm{t}} x_\alpha b$, $c_{\beta_1} = c_2 m_{\mathrm{t}} c_{m\beta} + c_1 m_{\mathrm{t}} x_\alpha b c_{l\beta}$, $c_{\gamma_1} = c_1 m_{\mathrm{t}} x_\alpha b c_{l\gamma} + c_2 m_{\mathrm{t}} c_{m\gamma}$, $c_{\alpha_2} = -c_2 m_{\mathrm{t}} x_\alpha b c_{m\alpha} - c_1 I_\alpha c_{l\alpha}$, $c_{\alpha_{21}} = m_{\mathrm{t}} x_\alpha b k_{\alpha_2}$, $c_{\dot{\alpha}_2} = -c_2 m_{\mathrm{t}} x_\alpha b c_{m\alpha} (0.5 - a) \frac{b}{U} - c_1 I_\alpha c_{l\alpha} (0.5 - a) \frac{b}{U} + c_{\alpha 0} m_{\mathrm{t}} x_\alpha b$, $c_{\dot{\alpha}_{21}} = m_{\mathrm{t}} x_\alpha b c_{\alpha 2}$, $c_{h_{21}} = -k_{h2} I_\alpha$, $c_{\dot{h}_2} = -c_2 m_{\mathrm{t}} x_\alpha c_{m\alpha} \frac{b}{U} - c_h I_\alpha - c_1 I_\alpha c_{l\alpha} \frac{1}{U}$, $c_{\beta_2} = -c_2 m_{\mathrm{t}} x_\alpha b c_{m\beta} - c_1 I_\alpha c_{l\beta}$, $c_{\gamma_2} = -c_2 m_{\mathrm{t}} x_\alpha b c_{m\gamma} - c_1 I_\alpha c_{l\gamma}$, $c_1 = \rho U^2 b s_p$, and $c_2 = \rho U^2 b^2 s_p$.

Since the sign of k_{h2} is unknown, the sign of the control coefficient $\ell_2(\cdot)$ is unknown and cannot be used in the control design. Taking the same structural parameters as [39] gives the values of model parameters used for simulation in Table I. Let the reference signal be $y_{\mathrm{r}}(t) = \sin(0.5t) + \sin(t)$. The initial state values are chosen as $\chi_1(0) = 3.5$, $\chi_2(0) = -1.5$, $\chi_3(0) = -2.5$ and $\chi_4(0) = -1.5$. The design parameters are chosen to be: $\varrho_1 = 1.25$, $\varrho_2 = 1.75$, $\varrho_3 = \varrho_4 = 5$, $\delta_1 = 0.75$, $\delta_2 = 0.5$, $\delta_3 = 0.35$, $\delta_4 = 0.9$, $\rho_{1,0} = \rho_{2,0} = \rho_{3,0} = \rho_{4,0} = 5$, $\rho_{1,\infty} = 0.1$, $\rho_{2,\infty} = 0.85$, $\rho_{3,\infty} = 0.5$, $\rho_{4,\infty} = 0.75$, $\kappa_1 = 1.25$, $\kappa_2 = 0.75$, $\kappa_3 = \kappa_4 = 0.5$. The parameters and initial conditions of Nussbaum functions are $\mu = 0.25$ and $s_1(0) = s_2(0) = s_3(0) = s_4(0) = 0$, respectively.

The simulation results are shown in Figs. 3 and 4. In particular, Fig. 3 (a) and (b) show that system output y tracks the reference signal y_{r} with bounded tracking error and that the output tracking error e_1 evolves within the prescribed bounds $(-\rho_1, \rho_1)$ in spite of the unknown control coefficient $\ell_2(\cdot)$. Fig. 3 (c) and (d) indicate the boundedness of the control signal u and the state variables χ_2, χ_3, and χ_4. Fig. 4 (a) and (b) show the boundedness of s_1, s_2, s_3, s_4, $\mathscr{N}(s_1)$, $\mathscr{N}(s_2)$, $\mathscr{N}(s_3)$, and $\mathscr{N}(s_4)$.

To investigate the influence of the parameter δ_i, $i = 1, \ldots, 4$, on the closed-loop response, we carry out the simulation based on three different sets of values of δ_i: Case 1: $\delta_1 = 0.15$, $\delta_2 = 0.2$, $\delta_3 = 0.25$, $\delta_4 = 0.3$; Case 2: $\delta_1 = 0.25$, $\delta_2 = 0.3$, $\delta_3 = 0.35$, $\delta_4 = 0.4$; Case 3: $\delta_1 = 0.35$, $\delta_2 = 0.45$, $\delta_3 = 0.55$, $\delta_4 = 0.6$. The trajectories

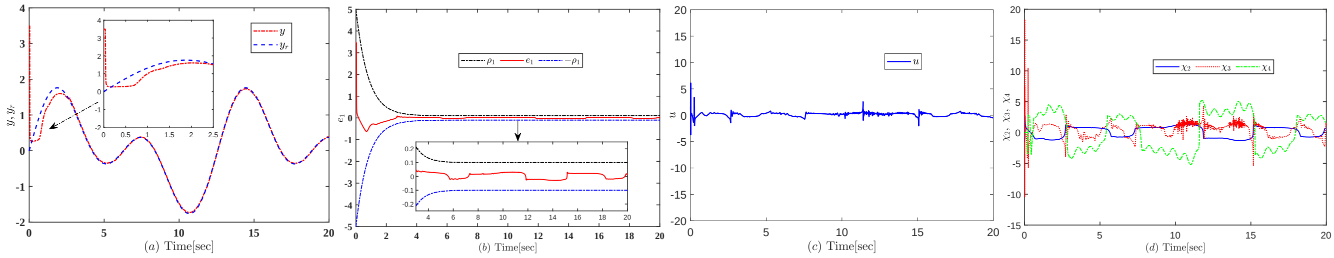

Figure 3. (a): Evolution of y and y_r; (b): Evolution of the tracking error e_1; (c): Evolution of the control input signal u; (d): Evolution of the state variables χ_2, χ_3, and χ_4.

of the adaptation parameters s_i are depicted in Fig. 5, which validate the boundedness of s_i for different δ_i, $i = 1, \ldots, 4$.

Table 1
The values of model parameters

Coefficient	Value	Coefficient	Value
$c_{\bar{\alpha}_1}$	0.7835	$c_{\alpha_{11}}$	−1.5616
$c_{\dot{\alpha}_{11}}$	−7.6423	$c_{\dot{h}_1}$	2.6583
c_{γ_1}	0.7256	c_{α_2}	−5.8731
$c_{\dot{\alpha}_2}$	−3.2567	$c_{\dot{\alpha}_{21}}$	1.2548
$c_{\dot{h}_2}$	−8.2431	c_{α_1}	0.5717
$c_{\dot{\alpha}_1}$	4.9527	c_{β_1}	0.5394
$c_{\alpha_{21}}$	2.2495	$c_{h_{21}}$	−0.6724
c_{α_0}	-1.0395	c_{α_2}	6.7242
k_{h_0}	2.3985	k_{h_1}	−4.7592
k_{h_2}	3.6937	k_{α_0}	−2.0593

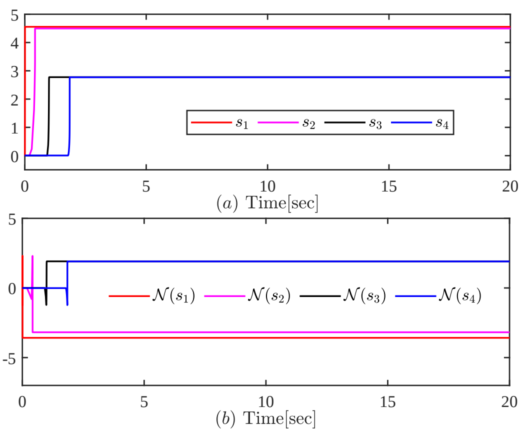

Figure 4. (a): Evolution of s_1, s_2, s_3, and s_4; (b): Evolution of $\mathscr{N}(s_1)$, $\mathscr{N}(s_2)$, $\mathscr{N}(s_3)$, and $\mathscr{N}(s_4)$.

To show the advantages of the proposed method in handling time-varying unknown control directions, two situations are considered: the proposed method with a type A Nussbaum function $\mathscr{N}(s) = \sin(3\pi s)s^2$ and with a type B Nussbaum function $\mathscr{N}(s) = \cos\left(\frac{\pi s}{2}\right)\exp(0.25s)$. The simulation results for a type B Nussbaum function have been already shown in Fig. 4, while the simulation results for a type A Nussbaum function

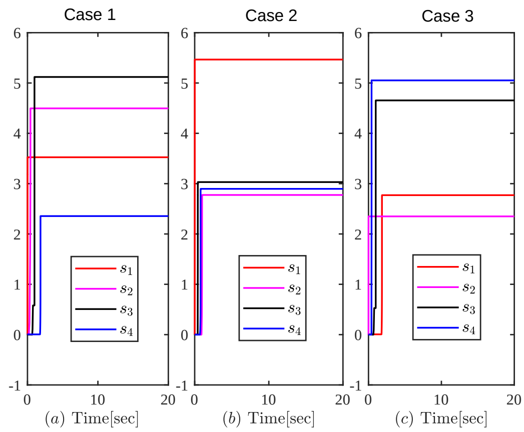

Figure 5. Evolution of s_1, s_2, s_3, and s_4 under three cases.

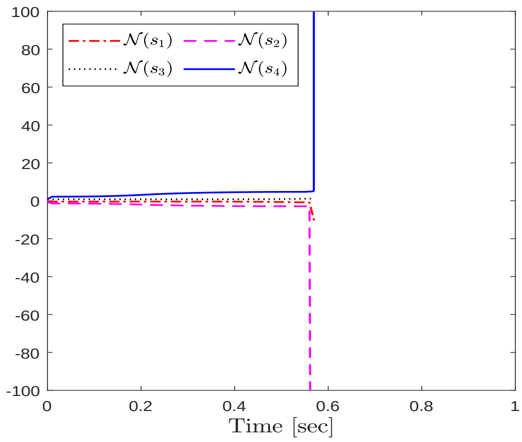

Figure 6. Evolution of $\mathscr{N}(s_1)$, $\mathscr{N}(s_2)$, $\mathscr{N}(s_3)$, and $\mathscr{N}(s_4)$ for a type A Nussbaum function.

are shown in Fig. 6, from which it can be seen that type A Nussbaum function (thought for fixed control coefficients) may fail to stabilize the system if the coefficients are not fixed anymore. In addition, the values of controller gains of two situations are quantified via several performance indices: integral absolute value (IAV) $\left[\int_0^T \sum_{i=1}^4 |\mathscr{N}_i'(t)|\mathrm{d}t\right]$, integral time absolute value (ITAV) $\left[\int_0^T \sum_{i=1}^4 t|\mathscr{N}_i'(t)|\mathrm{d}t\right]$, and root mean square value (RMSV) $\left[\frac{1}{T}\int_0^T \sum_{i=1}^4 \mathscr{N}'^2_i(t)\mathrm{d}t\right]^{\frac{1}{2}}$, where

Table 2
Performance indices with two types of Nussbaum functions.

	Type B	Type A
IAV	**110.71**	$\to\infty$
ITAV	**1296.74**	$\to\infty$
RMSV	**4.38**	$\to\infty$

$\mathscr{N}_i' = \varrho_i \mathscr{N}(s_i(\cdot))\mathfrak{T}_i(\cdot)$, $i = 1, \ldots, 4$ represents the value of controller gain. The calculation results are summarized in Table 2, which validates the advantages of our proposed method.

6 Conclusions

In the context of prescribed-performance control (PPC) for high-power dynamics with time-varying unknown control coefficients, this work has shown that only some particular type B Nussbaum functions can be used for handling time-varying unknown control coefficients in high-power systems. Then, a novel switching Nussbaum conditional inequality was designed to avoid high gains, by letting the switching gain increases only when the tracking error is close to violating the performance bounds. An interesting topic deserving future investigation is PPC with time-varying unknown control coefficients and positive-odd rational powers, which contains positive-odd integer powers as a special case.

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7 Appendix

Proof of Proposition 1: We first define some quantities as follows,

$$\begin{aligned} p_{\lambda,r} &= \int_0^1 \left[2^{\left(\lambda^3+\frac{1}{3}\right)\lambda}\sin\left(s\pi\right)\right]^r ds \\ &= 2^{r\left(\lambda^3+\frac{1}{3}\right)\lambda}\int_0^1 \sin^r\left(s\pi\right)ds = 2^{r\left(\lambda^4+\frac{1}{3}\lambda\right)}\alpha_r \end{aligned}$$

and

$$\begin{aligned} q_{\lambda,r} &= \int_1^{2^\lambda}\left[2^{\lambda^4}\sin\left(\frac{s-1}{2^\lambda-1}\pi\right)\right]^r ds = 2^{r\lambda^4}\int_1^{2^\lambda}\sin^r\left(\frac{s-1}{2^\lambda-1}\pi\right)ds \\ &= 2^{r\lambda^4}(2^\lambda-1)\int_0^1 \sin^r\left(s\pi\right)ds = 2^{r\lambda^4}(2^\lambda-1)\alpha_r \end{aligned}$$

for $\alpha_r = \int_0^1 \sin^r\left(s\pi\right)ds$. In accordance with Definition 2, the proof is divided into three parts.

(i) For any $y \geq 0$, there exists $\lambda \in \mathbb{N}^+$ such that $y \in [2^\lambda - 2, 2^{\lambda+1}-2)$. As a result, it holds that

$$\begin{aligned} \frac{1}{y}\int_0^y \mathscr{N}_-(s)ds \geq& \frac{1}{2^{\lambda+1}-2}\int_0^{2^\lambda-1}\mathscr{N}_-(s)ds \\ =& \frac{\sum\limits_{k=1}^{\lambda-1} q_{k,1}}{2^{\lambda+1}-2} = \frac{\sum\limits_{k=1}^{\lambda-1} 2^{k^4}(2^k-1)\alpha_1}{2^{\lambda+1}-2} \geq 2^\lambda\alpha_1 \end{aligned}$$

for $\lambda \geq 3$. The fact that $\lambda \to +\infty$ as $y \to +\infty$ implies $\lim_{y\to+\infty}\frac{1}{y}\int_0^y \mathscr{N}_-(s)ds = +\infty$.

(ii) Note the following calculation, with $y = 2^\lambda - 1$,

$$\frac{\int_0^y \mathscr{N}_+(s)ds}{\int_0^y \mathscr{N}_-(s)ds} = \frac{\sum_{k=1}^{\lambda} p_{k,1}}{\sum_{k=1}^{\lambda-1} q_{k,1}} = \frac{\sum_{k=1}^{\lambda} 2^{k^4+\frac{1}{3}k}}{\sum_{k=1}^{\lambda-1} 2^{k^4}\left(2^k-1\right)}. \quad (35)$$

It follows from the Stolz-Cesaro Theorem [40, Sect. 3.17, pp. 85, Theorem 1.22] that

$$\lim_{\lambda\to+\infty}\frac{\sum_{k=1}^{\lambda} 2^{k^4+\frac{1}{3}k}}{\sum_{k=1}^{\lambda-1} 2^{k^4}\left(2^k-1\right)} = \lim_{\lambda\to+\infty}\frac{2^{\lambda^4+\frac{1}{3}\lambda}}{2^{(\lambda-1)^4+\lambda-1}} = +\infty$$

which, together with (35), implies $\lim\limits_{y\to+\infty}\sup\frac{\int_0^y \mathscr{N}_+(s)ds}{\int_0^y \mathscr{N}_-(s)ds} = +\infty$.

The results $\lim\limits_{y\to+\infty}\frac{1}{y}\int\limits_0^y \mathscr{N}_+(s)ds = +\infty$ and $\lim\limits_{y\to+\infty}\sup\frac{\int_0^y \mathscr{N}_-(s)ds}{\int_0^y \mathscr{N}_+(s)ds} = +\infty$ can be proved in a similar way and are omitted. According to Definition 2, $\mathscr{N}(\cdot)$ is a type B Nussbaum function.

(iii) For any $y \geq 0$, there exists $\lambda \in \mathbb{N}^+$ such that $y \in [2^\lambda - 2, 2^{\lambda+1}-2)$. According to the definition of $\mathscr{N}^r$, we have

$$\frac{\int_0^y \mathscr{N}_-^r(s)ds}{\int_0^y \mathscr{N}_+^r(s)ds} \leq \frac{\int_0^{2^\lambda-2} \mathscr{N}_-^r(s)ds}{\int_0^{2^\lambda-2} \mathscr{N}_+^r(s)ds} \text{ or } \frac{\int_0^{2^{\lambda+1}-2} \mathscr{N}_-^r(s)ds}{\int_0^{2^{\lambda+1}-2} \mathscr{N}_+^r(s)ds}.$$

The following calculation

$$\begin{aligned} &\lim_{y\to+\infty}\sup\frac{\int_0^y \mathscr{N}_-^r(s)ds}{\int_0^y \mathscr{N}_+^r(s)ds} \leq \lim_{\lambda\to+\infty}\frac{\int_0^{2^{\lambda+1}-2} \mathscr{N}_-^r(s)ds}{\int_0^{2^{\lambda+1}-2} \mathscr{N}_+^r(s)ds} = \\ &\leq \lim_{\lambda\to+\infty}\frac{\sum_{k=1}^{\lambda} q_{k,r}}{\sum_{k=1}^{\lambda-1} p_{k,r}} = \lim_{\lambda\to+\infty}\frac{\sum_{k=1}^{\lambda} 2^{rk^4}\left(2^k-1\right)}{\sum_{k=1}^{\lambda} 2^{r\left(k^4+\frac{1}{3}k\right)}} \\ &= \lim_{\lambda\to+\infty}\left(2^{\lambda-\frac{r\lambda}{3}} - 2^{-\frac{r\lambda}{3}}\right) = \begin{cases} +\infty, & r=1; \\ 1, & r=3; \\ 0, & r>3. \end{cases} \end{aligned}$$

shows the violation of Definition 2. Thus, one can conclude that $\mathscr{N}^r(\cdot)$ is not a type B Nussbaum function. This completes the proof. ■

Proof of Proposition 2: According to [20], $\mathscr{N}^r(s)$ is a type B Nussbaum function for $r = 1$. So the remaining task is to show that statement still holds for $r \geq 3$. By the Darboux-Stieltjes integral property [40, Sect. 6.12, pp. 257, Theorem 1.7, (h)], one has, for any $a \geq 0$, it holds that

$$\begin{aligned}
&\exp\left(r\mu a^2\right)\alpha_r \leq \int_a^{a+1} \exp\left(r\mu s^2\right) \left|\cos^r\left(\frac{\pi s}{2}\right)\right| ds \\
&= \exp\left(r\mu \bar{s}^2\right) \int_a^{a+1} \left|\cos^r\left(\frac{\pi s}{2}\right)\right| ds \leq \exp\left(r\mu(a+1)^2\right)\alpha_r
\end{aligned} \tag{36}$$

for some $\bar{s} \in (a, a+1)$ and $\alpha_r = \int_0^1 \cos^r\left(\frac{\pi s}{2}\right) ds$, which is used in the remaining proof. In accordance with Definition 2, the proof is divided into two parts.

(i) For any $y \geq 0$, there exists $\lambda \in \mathbb{N}$ such that $y \in [4\lambda - 3, 4\lambda + 1)$, where $\mathbb{N}$ is the set of integers. As a result, one has

$$\begin{aligned}
\frac{1}{y}\int_0^y \mathscr{N}_+^r(s)ds &> \frac{1}{4\lambda+1}\int_0^{4\lambda-1} \mathscr{N}_+^r(s)ds \\
&> \frac{1}{4\lambda+1}\sum_{k=1}^{\lambda-1}\int_{4k-1}^{4k+1} \exp\left(r\mu s^2\right)\cos^r\left(\frac{\pi s}{2}\right)ds.
\end{aligned}$$

By (36), we can arrive at

$$\int_{4k-1}^{4k+1} \exp\left(r\mu s^2\right)\cos^r\left(\frac{\pi s}{2}\right)ds \geq 2\alpha_r \exp\left(r\mu(4k-1)^2\right)$$

and hence

$$\lim_{y\to+\infty} \frac{1}{y}\int_0^y \mathscr{N}_+^r(s)ds \geq \lim_{\lambda\to+\infty} \frac{2\alpha_r \sum_{k=1}^{\lambda-1} \exp\left(r\mu(4k-1)^2\right)}{4\lambda+1} = +\infty.$$

(ii) Note the following calculation, with $y = 4\lambda + 3$,

$$\begin{aligned}
\frac{\int_0^y \mathscr{N}_-^r(s)ds}{\int_0^y \mathscr{N}_+^r(s)ds} &= \frac{\int_0^{4\lambda+3} \mathscr{N}_-^r(s)ds}{\int_0^{4\lambda+3} \mathscr{N}_+^r(s)ds} \\
&> \frac{\sum_{k=0}^{\lambda}\int_{4k+1}^{4k+3} \exp\left(r\mu s^2\right)\left|\cos^r\left(\frac{\pi s}{2}\right)\right| ds}{\sum_{k=0}^{\lambda}\int_{4k-1}^{4k+1} \exp\left(r\mu s^2\right)\cos^r\left(\frac{\pi s}{2}\right)ds} \\
&\geq \frac{\sum_{k=0}^{\lambda}\left[\exp\left(r\mu(4k+1)^2\right) + \exp\left(r\mu(4k+2)^2\right)\right]\alpha_r}{\sum_{k=0}^{\lambda}\left[\exp\left(r\mu(4k)^2\right) + \exp\left(r\mu(4k+1)^2\right)\right]\alpha_r}.
\end{aligned}$$

It follows from Stolz-Cesaro Theorem [40] that

$$\lim_{\lambda\to+\infty} \frac{\int_0^{4\lambda+3} \mathscr{N}_-^r(s)ds}{\int_0^{4\lambda+3} \mathscr{N}_+^r(s)ds} \geq \lim_{\lambda\to+\infty} \frac{\exp\left(r\mu(4\lambda+2)^2\right)}{\exp\left(r\mu(4\lambda+1)^2\right)} = +\infty,$$

which implies $\lim_{y\to+\infty}\sup \frac{\int_0^y \mathscr{N}_-^r(s)ds}{\int_0^y \mathscr{N}_+^r(s)ds} = +\infty$.

The results $\lim_{y\to+\infty} \frac{1}{y}\int_0^y \mathscr{N}_-^r(s)ds = +\infty$ and $\lim_{y\to+\infty}\sup \frac{\int_0^y \mathscr{N}_+^r(s)ds}{\int_0^y \mathscr{N}_-^r(s)ds}$ $= +\infty$ can be proved similarly. According to Definition 2, $\mathscr{N}^r(\cdot)$ is a type B Nussbaum function. ■