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Stability bounds for fuzzy estimation and control – Part II: Output-feedback control

Zs. Lendek, R. Babuška, and B. De Schutter

***Abstract*— A large class of nonlinear systems can be well approximated by Takagi-Sugeno fuzzy models, for which methods and algorithms have been developed to analyze their stability and to design observers and controllers. However, results obtained for Takagi-Sugeno fuzzy models are in general not directly applicable to the original nonlinear system. In this paper, we investigate what conclusions can be drawn when an observer-based controller is designed for an approximate model and then applied to the original nonlinear system. In particular, we consider the case when the scheduling vector used in the membership functions of the observer depends on the states that have to be estimated. The results are illustrated using simulation examples.**

I. Introduction

A large class of nonlinear functions can be exactly represented or well-approximated by Takagi-Sugeno (TS) fuzzy models [23]. The TS fuzzy model consists of a rule-base. The antecedents partition a subset of the variables into fuzzy regions, while the consequent of each rule is in general a linear or affine model, valid locally in the corresponding region. Methods to derive an exact fuzzy representation of a nonlinear exist [22], but in many cases the local models obtained are not controllable and/or observable.

Therefore, in this paper we consider fuzzy models that approximate a given nonlinear system. Several methods exist to construct TS models such that they approximate a given nonlinear model to an arbitrary degree of accuracy [9], [19]. In this case, since the fuzzy model only approximates the original nonlinear system, the controller and/or observer designed for the fuzzy model may not perform as expected for the nonlinear system. For instance, a stabilizing controller designed for the fuzzy model may not stabilize the original nonlinear system.

For a class of nonlinear systems, when using control based on TS fuzzy models, this shortcoming has been circumvented by the use of robust controllers. Robust fuzzy control has attracted increased research interest in the last decade. Results range from fuzzy control of nonlinear systems in canonical forms [1], [3], [10], through control of fuzzy systems with parametric uncertainties [2], [4], [6], [11] to delay-dependent fuzzy systems [5], [12], [17]. Applications include control of robotic manipulators [13], [15], [16], magnetic bearing systems [7], [14], and vehicle lateral dynamics [8].

Although there is an impressive body of literature concerning robust fuzzy control, observer design and the contribution of the estimation error to stabilization using output-feedback is rarely discussed. In particular, the effect of the observer designed for the approximate model on the stability of the closed-loop system has not been investigated. This problem becomes even more complicated when the scheduling variables themselves have to be estimated. Therefore, in this paper, we investigate whether and when conclusions can be drawn about the performance of an observer-based controller. Both the observer and the controller are designed based on the approximate fuzzy model and then applied to the nonlinear system. In order to keep the computations simple, a common quadratic Lyapunov function is used.

The structure of the paper is as follows. Section II presents the models used and reviews some classic results for the stability of autonomous fuzzy systems. The stability analysis of the fuzzy model and the analysis of the fuzzy observer designed based on the approximate fuzzy model has been performed in the companion paper [20]. In this paper we continue by investigating the stabilization of the nonlinear system using a fuzzy state feedback controller in Section III and observer-based output-feedback controller in Section IV. The different cases are illustrated using examples in the corresponding sections. Section V concludes the paper.

II. Preliminaries

We consider the following nonlinear system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})\end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the vector of the state variables, $\boldsymbol{u}$ is the input vector, $\boldsymbol{y}$ is the measurement vector. We assume that the variables are defined on a compact set $\mathcal{C}_{\boldsymbol{xuy}}$, i.e., $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \in \mathcal{C}_{\boldsymbol{xuy}}$. A TS fuzzy approximation of this system can be obtained as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}^\diamond(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\boldsymbol{u})\\ \boldsymbol{y} &= \boldsymbol{h}^\diamond(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i)\end{aligned} \tag{2}$$

so that the approximation errors $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^\diamond$ and $\overline{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}^\diamond$ satisfy

$$\begin{aligned}\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| &\le \sigma_f + \delta_f\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}\\ \|\overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})\| &\le \sigma_h + \delta_h\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}\end{aligned} \tag{3}$$

where σ_f, σ_h, δ_f, and δ_h are nonnegative finite constants, and $\mathcal{C}_{\boldsymbol{xu}} = \{(\boldsymbol{x}, \boldsymbol{u}) | \exists \boldsymbol{y} \text{ s.t. } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \in \mathcal{C}_{\boldsymbol{xuy}}\}$. In (2), A_i, B_i, C_i, D_i, and d_i, $i = 1, 2, \ldots, m$ represent the matrices and

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biases of the ith local linear model and w_i, $i = 1, 2, \ldots, m$ are the corresponding membership functions, which depend on the scheduling variable $\boldsymbol{x}$.

Throughout the paper it is assumed that the membership functions are normalized, i.e., $w_i(\boldsymbol{x}) \geq 0$, $\sum_{i=1}^{m} w_i(\boldsymbol{x}) = 1$, $\forall(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}$. I and 0, respectively, denote the identity and the zero matrices of the appropriate dimensions, $\mathcal{H}(A)$ represents the Hermitian of the matrix A, i.e., $\mathcal{H}(A) = A + A^T$, and $\|.\|$ denotes the Euclidean norm for vectors and the induced norm for matrices.

The nonlinear system (1) is now expressed as an uncertain TS system, given as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) + \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{x})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i) + \overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u}) \end{aligned} \tag{4}$$

where the uncertainties $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ satisfy (3).

Note that the approximation error on a compact set of variables always satisfies

$$\begin{aligned} \|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma_f \\ \|\overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})\| &\leq \sigma_h \end{aligned} \tag{5}$$

for some σ_f and σ_h. However, as will be shown in the sequel, by using (3) whenever possible, less conservative conditions can be obtained.

Remark: In the robust fuzzy control literature, for uncertain fuzzy systems in general the form

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{x})\Big[(A_i + \Delta A_i)\boldsymbol{x} + (B_i + \Delta B_i)\boldsymbol{u}\Big]$$

is used, for which asymptotic stability can be obtained. In order to be able to draw more general conclusions, in this paper the TS approximation (4) is used.

Our results are based on the following conditions [24] for the stability of autonomous fuzzy systems:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z})A_i\boldsymbol{x} \tag{6}$$

where A_i, $i = 1, 2, \ldots, m$ represents the ith local linear model, w_i is the corresponding normalized membership function, and $\boldsymbol{z}$ the vector of the scheduling variables, which may depend on the states, input, output, or other measured exogenous variables.

Theorem 1: [24] System (6) *is exponentially stable if there exists* $P = P^T > 0$ *so that*

$$\mathcal{H}(PA_i) < 0 \tag{7}$$

for $i = 1, 2, \ldots, m$. □

Controller and observer design for fuzzy systems of the form (2) often leads to establishing the negative definiteness of double summations of the form $\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\Upsilon_{ij}$, with Υ_{ij}, $i, j = 1, 2, \ldots m$ matrices of appropriate dimensions. In this paper we use the following relaxations for such sums [24]:

Theorem 2: Let Υ_{ij} *be matrices of proper dimensions. Then,*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\Upsilon_{ij} < 0 \tag{8}$$

holds, if

$$\begin{aligned} &\Upsilon_{ii} < 0 \qquad \text{for } i = 1, 2, \ldots, m, \\ &\frac{1}{2}(\Upsilon_{ij} + \Upsilon_{ji}) < 0, \qquad \text{for } i, j = 1, 2, \ldots, m,\ i \neq j \end{aligned} \tag{9}$$

Note that similar, although more complex results can also be derived using other types of Lyapunov functions, as long as the derived conditions ensure the exponential stability of the TS system.

In the companion paper [20], we have already presented stability analysis of autonomous TS systems and the conclusions that may be drawn regarding the nonlinear system that has been approximated by it. Therefore, we do not repeat those conclusions here, and instead refer the interested reader to [20] for further details.

III. Stabilization using full state-feedback

Development of sufficient conditions for the stabilization using full state-feedback of uncertain TS fuzzy systems has received increasing interest in the last years [4], [6], [7], [11], in particular for discrete-time systems. In this paper, we consider continuous-time TS systems. Instead of developing conditions to design controllers that asymptotically stabilize the system, we investigate what conclusions regarding the original nonlinear system can be drawn if a controller has already been designed for its fuzzy approximation. Therefore, consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{10}$$

that is approximated by the TS system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}^{\diamond}(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) \tag{11}$$

so that the approximation error $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^{\diamond}$ satisfies

$$\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \leq \sigma_f + \delta_f\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}} \tag{12}$$

with σ_f and δ_f being nonnegative finite constants.

Although for observer design the state transition model may contain affine terms (see [20]), for stabilization, the nonlinear system has to be approximated by a fuzzy model of the form (11), i.e., the local models may not be affine and the membership functions may not depend on the control input $\boldsymbol{u}$. This is firstly because stabilization to zero of affine fuzzy systems using a classical fuzzy state-feedback can only be performed if the affine term is compensated for in each rule. Secondly, if the membership functions depend on the control input, when actually computing the input, an implicit equation has to be solved.

Using a classical fuzzy state-feedback

$$\boldsymbol{u} = \sum_{i=1}^{m} w_i(\boldsymbol{x})K_i\boldsymbol{x}$$

we have the closed-loop fuzzy system:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\boldsymbol{x})(A_i + B_iK_j)\boldsymbol{x} \tag{13}$$

and the dynamics of the closed-loop nonlinear system can be described as

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\boldsymbol{x})(A_i + B_iK_j)\boldsymbol{x} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \tag{14}$$

With a common quadratic Lyapunov function, the TS system (13) is globally exponentially stable, according to Theorem 2, if there exist $P = P^T > 0$, $Q = Q^T > 0$ so that

$$\begin{aligned}&\mathcal{H}(P(A_i + B_iK_i)) < -2Q\\&\mathcal{H}(P(A_i + B_iK_j) + P(A_j + B_jK_i)) < -4Q\\&\text{for all } j,\, i = 1,\,2,\,\ldots,\,m,\; i \neq j\end{aligned} \tag{15}$$

With the same Lyapunov function applied to the original nonlinear system (1), we obtain:

$$\begin{aligned}\dot{V} &= \boldsymbol{x}^T\mathcal{H}(P(\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\boldsymbol{x})G_{ij}\boldsymbol{x} + \overline{\boldsymbol{f}}(\boldsymbol{x})))\\&\leq -2(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)(1-\theta)\|\boldsymbol{x}\|^2\\&\quad - 2\|\boldsymbol{x}\|(\theta(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)\|\boldsymbol{x}\| - \lambda_{\max}(P)\sigma_f)\end{aligned}$$

with $\theta \in (0,1)$ arbitrarily chosen and $G_{ij} = A_i + B_iK_j$.

By analyzing the expression of $\dot{V}$, the following cases can be distinguished:

1) ($\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f < 0$) or ($\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f > 0$): no conclusion can be drawn;
2) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f = 0$: if the membership functions are sufficiently smooth, and $\boldsymbol{x} = 0$ is the only equilibrium point, based on LaSalle's invariance principle and Barbalat's lemma [18], $\boldsymbol{x} = 0$ is a globally asymptotically stable equilibrium point of the nonlinear system (14). Such results are in general enforced when adaptive fuzzy controllers are designed.
3) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and $\sigma_f = 0$: the nonlinear system (14) has a globally exponentially stable equilibrium point in $\boldsymbol{x} = 0$. Note that this result can only be obtained if the approximation error is Lipschitz continuous in the states.
4) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and $\sigma_f > 0$: the states of the nonlinear system (14) are uniformly ultimately bounded by

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f}\frac{\sigma_f}{\theta}. \tag{16}$$

It is important to note that in robust fuzzy control the affine term in (12) is in general considered to be an external disturbance affecting the system, and not a model mismatch, i.e, uncertainty is presumed to affect only the matrices A_i and B_i, $i = 1,\,2,\,\ldots,\,m$. Nevertheless, even if the disturbance is due to model mismatch, robust controllers that are able to attenuate its effect can be designed.

If the controller has already been designed using (15), only the above conditions can be verified. However, if the controller is to be designed, then, in order to obtain a bound as small as possible one can also solve the multi-objective optimization problem:

$$\begin{aligned}&\text{maximize } \lambda_{\min}(Q),\ \lambda_{\min}(P),\\&\text{minimize } \lambda_{\max}(P),\\&\text{subject to}\\&P = P^T > 0\\&Q = Q^T > 0\\&\mathcal{H}(P(A_i + B_iK_i)) \leq -2Q,\ \ i = 1,\,2,\,\ldots,\,m\\&\mathcal{H}(P(A_i + B_iK_j) + P(A_j + B_jK_i)) \leq -4Q\\&\ j, i = 1,\,2,\,\ldots,\,m\end{aligned} \tag{17}$$

Example 1: Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1.1 & x_1^2 + 0.1\\ -x_1 - 1 & -3 - x_2^2\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}1\\0\end{pmatrix}\boldsymbol{u} \tag{18}$$

with $x_1,\, x_2 \in [-1,\,1]$. This system is unstable.

A TS approximation of the system (18) is obtained using the approach in [19]. Normalized triangular membership functions are chosen, that attain their maximum in the points defined by $\{(x_1,\,x_2)|x_1,\,x_2 \in \{-1,0,1\}\}$. The TS system can be written as:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + Bu) \tag{19}$$

The approximation errors can be written as $\|\overline{\boldsymbol{f}}\| \leq \sigma_f + \delta_f\|\boldsymbol{x}\| = 0.407\alpha + 0.48(1-\alpha)\|\boldsymbol{x}\|$, with α arbitrarily chosen in $[0,\,1]$, and $\|\overline{\boldsymbol{h}}\| = \sigma_h + \delta_h\|\boldsymbol{x}\| = 0$.

By simply solving the feasibility problem[1]

Find $P = P^T > 0$, $Q = Q^T > 0$, s.t. (15) *is satisfied*

one obtains $P = \begin{pmatrix} 6.5 & 0.33\\ 0.33 & 0.38\end{pmatrix}$, $\lambda_{\min}(P) = 0.36$, $\lambda_{\max}(P) = 6.52$, $Q = I$. With these results, the following cases can be distinguished:

1) if α is chosen such that $\alpha < 0.69$, then we have $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and therefore no conclusion can be drawn
2) for $\alpha > 0.69$, we have Case 4, i.e., the states of the controlled nonlinear system (18), using the controller designed for the fuzzy system (19) are ultimately uniformly bounded by

$$\begin{aligned}\gamma &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta}\frac{\sigma}{\theta}\\&= \frac{11.3\alpha}{(1 - 3.13(1-\alpha))\theta} < 11.3\end{aligned}$$

with $\alpha \in [0.69,\,1]$ and $\theta \in (0,\,1)$.

[1]For solving the LMI problems in this paper, the *sedumi* solver of Yalmip [21] has been used.

Solving (17), i.e., minimizing[2] $\lambda_{\max}(P)$ and maximizing $\lambda_{\min}(Q)$ and $\lambda_{\min}(P)$, one obtains: $P = \begin{pmatrix} 0.20 & 0.002 \\ 0.002 & 0.17 \end{pmatrix}$, $\lambda_{\min}(P) = 0.17$, $\lambda_{\max}(P) = 0.20$ and $Q = I$.

With these values, depending on the choice of α, we have the following cases:

1) for $\alpha = 0$ we have $\sigma_f = 0$ and $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and therefore the states of the nonlinear system converge exponentially to 0
2) for $\alpha = 1$, i.e., when a constant approximation error is considered, the states of the nonlinear system (18) are uniformly ultimately bounded by $\gamma = \frac{0.088}{\theta}$, with $\theta \in (0,\ 1)$, i.e., $\gamma < 0.088$.
3) otherwise, we obtain that the states are uniformly ultimately bounded by

$$\begin{aligned}\gamma &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta} \frac{\sigma}{\theta} \\ &= \frac{0.088\alpha}{(1 - 0.48(1-\alpha))\theta}\end{aligned}$$

with $\theta \in (0,\ 1)$ and $\alpha \in (0,\ 1)$.

As illustrated above, by solving the optimization problem together with the design problem, not only a lower bound, but even exponential convergence of the nonlinear system can be obtained. □

IV. Output-feedback control

Although output-feedback control is often considered in robust fuzzy control, it is in general assumed that the controller is able to compensate for or attenuate the disturbance resulting from the mismatch between the model used by the observer and the true system, without explicitly analyzing this model mismatch. In this section, although we do not design robust controllers, we analyze the disturbance due to the mismatch and investigate what guarantees can be given in this case.

Note that also in this case, the membership function cannot depend on the control input, and the state transition function cannot have an affine term, i.e., the same restrictions as for Section III apply. Therefore, the approximation considered is (2), with the approximation errors bounded as (3).

The observer is of the form

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(C_i\widehat{\boldsymbol{x}} + D_i\boldsymbol{u} + d_i)\end{aligned} \tag{20}$$

and the controller used is

$$\boldsymbol{u} = \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})K_i\widehat{\boldsymbol{x}} \tag{21}$$

[2]For solving this problem, a single objective function that was the linear combination of the objectives in (17) has been optimized.

The estimation error for the nonlinear system can be derived as:

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}^{\diamond}(\widehat{\boldsymbol{x}},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\boldsymbol{e} - L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ &\quad + \sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u}) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\boldsymbol{e} - L_i \\ &\qquad \cdot (\sum_{j=1}^{m} w_j(\boldsymbol{x})(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}) \\ &\quad - \sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}})(C_j\widehat{\boldsymbol{x}} + D_j\boldsymbol{u} + d_j))) \\ &\quad + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\boldsymbol{e} - L_i(\sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}})C_j\boldsymbol{e} \\ &\quad + \sum_{j=1}^{m}(w_j(\boldsymbol{x}) - w_j(\widehat{\boldsymbol{x}}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) \\ &\quad + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}))) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}})w_j(\widehat{\boldsymbol{x}})(A_i - L_iC_j)\boldsymbol{e} \\ &\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\end{aligned} \tag{22}$$

with

$$\begin{aligned}\Delta_{wf} &= \sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u}) \\ \Delta_{wh} &= \sum_{j=1}^{m}(w_j(\boldsymbol{x}) - w_j(\widehat{\boldsymbol{x}}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j)\end{aligned}$$

Note that since the goal is also stabilization, in this case the bounds on $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ can contain a term that is Lipschitz in $\boldsymbol{x}$. Moreover, one could also use bounds on Δ_{wf} and Δ_{wh} such as

$$\begin{aligned}\|\Delta_{wf}\| &\leq \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| + \eta_{wf}\|\boldsymbol{x}\| \\ \|\Delta_{wh}\| &\leq \sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| + \eta_{wh}\|\boldsymbol{x}\|\end{aligned} \tag{23}$$

However, for the simplicity of the computations, in this paper the following bounds are assumed:

$$\begin{aligned}\|\Delta_{wf}\| &\leq \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\ \|\Delta_{wh}\| &\leq \sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\|\end{aligned} \tag{24}$$

Then, in the worst case, the following bound can be derived:

$$
\begin{aligned}
&\| -\sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}}) L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x}, \boldsymbol{u})) \\
&\quad + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \\
&\qquad \le \max_i \|L_i\| (\sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| + \sigma_h + \delta_h\|\boldsymbol{x}\|) \\
&\qquad\quad + \sigma_f + \delta_f\|\boldsymbol{x}\| + \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\
&\qquad \le \sigma_e + \delta_e\|\boldsymbol{e}\| + \eta_e\|\boldsymbol{x}\|
\end{aligned} \tag{25}
$$

with

$$
\begin{aligned}
\sigma_e &= \max_i \|L_i\|(\sigma_{wh} + \sigma_h) + \sigma_f + \sigma_{wf} \\
\delta_e &= \max_i \|L_i\|\delta_{wh} + \delta_{wf} \\
\eta_e &= \max_i \|L_i\|\delta_h + \delta_f
\end{aligned} \tag{26}
$$

In fact:

$$
\begin{aligned}
\dot{\boldsymbol{e}} &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}}) w_j(\widehat{\boldsymbol{x}})(A_i - L_i C_j)\boldsymbol{e} + \Delta_e \\
\|\Delta_e\| &\le \sigma_e + \delta_e\|\boldsymbol{e}\| + \eta_e\|\boldsymbol{x}\|
\end{aligned} \tag{27}
$$

Second, the closed-loop dynamics using the estimate-based control law is:

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i \sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}}) K_j \widehat{\boldsymbol{x}}) \\
&\quad + \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u}) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x}) w_j(\widehat{\boldsymbol{x}})[(A_i + B_i K_j)\boldsymbol{x} + B_i K_j \boldsymbol{e}] \\
&\quad + \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})
\end{aligned} \tag{28}
$$

with

$$
\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \le \sigma_f + \delta_f\|\boldsymbol{x}\| \tag{29}
$$

Combining the dynamics of the estimation error and the state, we get

$$
\begin{aligned}
\begin{pmatrix} \dot{\boldsymbol{e}} \\ \dot{\boldsymbol{x}} \end{pmatrix} &= \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} w_i(\widehat{\boldsymbol{x}}) w_j(\boldsymbol{x}) w_k(\widehat{\boldsymbol{x}}) \\
&\begin{pmatrix} A_i - L_i C_k & 0 \\ K_k & A_j + B_j K_k \end{pmatrix} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{x} \end{pmatrix} + \Delta
\end{aligned} \tag{30}
$$

with

$$
\Delta = \begin{pmatrix} \Delta_e \\ \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u}) \end{pmatrix} \tag{31}
$$

Knowing that $\|\Delta_e\| \le \sigma_e + \delta_e\|\boldsymbol{e}\| + \eta_e\|\boldsymbol{x}\|$ and $\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \le \sigma_f + \delta_f\|\boldsymbol{x}\|$, we have

$$
\begin{aligned}
\|\Delta\| &\le \|\Delta_e\| + \|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \\
&\le \sigma_e + \sigma_f + \delta_e\|\boldsymbol{e}\| + (\eta_e + \delta_f)\|\boldsymbol{x}\| \\
&\le \sigma + \delta \left\| \begin{matrix} \boldsymbol{e} \\ \boldsymbol{x} \end{matrix} \right\|
\end{aligned}
$$

where $\sigma = \sigma_e + \sigma_f$ and $\delta = \sqrt{2}\max\{\delta_e,\, \eta_e + \delta_f\}$.

For the above bounds, the same cases can be distinguished as in the previous section. However, it has to be noted that firstly, Case 2) and Case 3) (see Section III) in practice will only be obtained if the fuzzy model is an exact representation of the nonlinear system and the membership functions do not depend on unmeasured variables. Secondly, the bound obtained in Case 4) is very conservative, and therefore in practical cases the applied output-feedback obtains better results than those that can be concluded based on this bound. Moreover, also due to the conservativeness of the result, the design of the output-feedback control such that some desired bounds are satisfied is not practical. However, the bounds can also be computed after designing the observer and controller, and therefore be used to establish guarantees for the closed-loop system.

The following example illustrates the computation of the bounds for output-feedback control:

Example 2: Consider the nonlinear system

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_2^2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{u} \\
\boldsymbol{y} &= [1\ 0]\boldsymbol{x}
\end{aligned} \tag{32}
$$

with $x_1,\, x_2 \in [-1,\, 1]$.

A TS approximation of this system is obtained as in Example 1, where the approximation errors are $\|\overline{\boldsymbol{f}}\| \le \sigma_f + \delta_f\|\boldsymbol{x}\| = 0.407\alpha + 0.48(1-\alpha)\|\boldsymbol{x}\|$, $\alpha \in [0,\, 1]$ and $\|\overline{\boldsymbol{h}}\| = 0$. With the same membership functions as in Example 1, we also have $\Delta_{wf} \le \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| = \beta \cdot 6.3 + (1-\beta)\cdot 6.3\|\boldsymbol{e}\|$, with $\beta \in [0,\, 1]$. Since the measurement matrix is common for all the rules, the equations can be simplified, and we have $\|\Delta_{wh}\| = 0$. Consequently, $\sigma_e = 0.407\alpha + 6.3\beta$, $\delta_e = 6.3(1-\beta)$, and $\eta_e = 0.48(1-\alpha)$, and $\sigma = 0.814\alpha + 6.3\beta$, and $\delta = \sqrt{2}\max\{6.3(1-\beta), 0.96(1-\alpha)\}$.

Solving the problem

Find $P = P^T > 0$, $Q = Q^T > 0$ such that

$$
\mathcal{H}\Big(P \begin{pmatrix} A_i - L_i C_k & 0 \\ K_k & A_j + B_j K_k \end{pmatrix}\Big) < -2Q
$$

for all $j, i = 1,\, 2,\, \ldots,\, m$

we obtain $P = \begin{pmatrix} 12.53 & 0 & 0 & 0 \\ 0 & 12.53 & 0 & 0 \\ 0 & 0 & 9.06 & 2.90 \\ 0 & 0 & 2.90 & 1.03 \end{pmatrix}$ and $Q = I$, $\lambda_{\min}(P) = 0.09$, and $\lambda_{\max}(P) = 12.53$. With these values, we have the bound on the state and estimation error

$$
\gamma = \frac{147.8(0.814\alpha + 6.3\beta)}{\theta(1 - \sqrt{2}\max\{6.3(1-\beta), 0.96(1-\alpha)\}} \tag{33}
$$

under the condition that

$$
1 - \sqrt{2}\max\{6.3(1-\beta), 0.96(1-\alpha)\} > 0
$$

and $\alpha, \beta, \theta \in (0,\, 1)$. It can be easily seen that this bound is very large, irrespective of the values chosen for α, β, such that (33) is satisfied. However, a large part of this bound is due to the observer-model error. For instance, consider the system

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_1^2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{u} \\
\boldsymbol{y} &= [1\ 0]\boldsymbol{x}
\end{aligned} \tag{34}
$$

with $x_1, x_2 \in [-1, 1]$. The difference with respect to the system (32) is that the $(2, 2)$ element of the matrix depends on x_1, instead of x_2. For this system, the membership functions will only depend on $x_1 = y$, i.e., on the measured variable. Therefore, in the membership functions of the observer, we can use its true value, and consequently $\Delta_{wf} = \Delta_{wh} = 0$. Moreover, a solution such that $\lambda_{\max}(P) = 1.05$, $\lambda_{\min}(P) = 0.73$, and $\lambda_{\min}(Q) = 1$ can also be obtained. With these values, the bound on the estimation error becomes

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta} \frac{\sigma}{\theta} = \frac{0.512\alpha}{(1 - 0.5(1 - \alpha))\theta}$$

with $\theta \in (0, 1)$ and $\alpha \in [0, 1]$. It can easily be seen that for $\alpha = 0$, this bound is actually 0, and therefore both the states of the nonlinear system and the estimation error converge to 0. □

V. Conclusions

In this paper we have investigated what stability guarantees can be obtained when a controller is designed for a fuzzy approximation of a nonlinear system and applied to the original nonlinear system. We have studied how the guarantees depend on the approximation error and on the mismatch between the observer-model and the true system. In our future research we will investigate whether the results can be improved by using other types of observers or controllers.

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References

[1] H. Allamehzadeh and J. Y. Cheung, "Robust fuzzy control with sliding mode property and inherent boundary layer," in *Proceedings of the American Control Conlerence*, Denver, Colorado, June 2003, pp. 4231–4236.

[2] S. Bai and S. Zhang, "Robust fuzzy control of uncertain nonlinear systems based on linear matrix inequalities," in *Proceedings of the 7th World Congress on Intelligent Control and Automation*, Chongqing, China, June 2008, pp. 4337–4341.

[3] R. Boukezzoula, S. Galichet, and L. Foulloy, "Robust fuzzy control for a class of nonlinear system using input-output linearization. Real-time implementation for a robot wrist." in *Proceedings of the 2001 International Conference on Control Applications*, Mexico City, Mexico, September 2001, pp. 311–316.

[4] M. Chadli and A. El Hajjaji, "Output robust stabilisation of uncertain Takagi-Sugeno model," in *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference*, Seville, Spain, December 2005, pp. 3393–3398.

[5] B. Chen and X. Liu, "Delay-dependent robust H_∞ control for TS fuzzy systems with time delay," *IEEE Transactions on Fuzzy Systems*, vol. 13, no. 4, pp. 544–556, 2005.

[6] B.-S. Chen, C.-S. Tseng, and H.-J. Uang, "Robustness design of nonlinear dynamic systems via fuzzy linear control," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 5, pp. 575–585, 1999.

[7] H. Du, N. Zhang, J. C. Ji, , and W. Gao, "Robust fuzzy control of an active magnetic bearing subject to voltage saturation," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 1, pp. 164–169, 2009.

[8] A. El Hajjaji, M. Chadli, M. Oudghiri, and O. Pagès, "Observer-based robust fuzzy control for vehicle lateral dynamics," in *Proceedings of the 2006 American Control Conference*, Minneapolis, Minnesota, June 2006, pp. 4664–4669.

[9] C. Fantuzzi and R. Rovatti, "On the approximation capabilities of the homogeneous Takagi-Sugeno model," in *Proceedings of the Fifth IEEE International Conference on Fuzzy Systems*, New Orleans, Louisiana, September 1996, pp. 1067–1072.

[10] M. B. Ghalia, "Robust model-based control of uncertain dynamical systems: A fuzzy set theory based approach," in *Proceedings of the 35th Conference on Decision and Control*, Kobe, Japan, December 1996, pp. 807–812.

[11] C.-Z. Gong, L. Li, and W. Wang, "Observer-based robust fuzzy control of nonlinear discrete systems with parametric uncertainties," in *Proceedings of the Third International Conference on Machine Learning and Cybernetics*, Shanghai, China, August 2004, pp. 367–371.

[12] J. Haibo, Y. Jianjiang, and Z. Caigen, "Robust fuzzy control of nonlinear delay systems subject to impulsive disturbance of input," in *Proceedings of the 26th Chinese Control Conference*, Zhangjiajie, Hunan, China, July 2007, pp. 289–293.

[13] C. Ham, Z. Qu, and R. Johnson, "Robust fuzzy control for robot manipulators," *IEE Proceedings–Control Theory and Applications*, vol. 147, no. 2, pp. 212–216, 2000.

[14] S.-K. Hong and R. Langari, "Robust fuzzy control of a magnetic bearing system subject to harmonic disturbances," *IEEE Transactions on Control Systems Technology*, vol. 8, no. 2, pp. 366–371, 2000.

[15] F.-Y. Hsu and L.-C. Fu, "Adaptive robust fuzzy control for robot manipulators," in *Proceedings of the IEEE International Conference on Robotics and Automation*, San Diego, CA, USA, May 1994, pp. 649–654.

[16] ——, "Nonlinear control of robot manipulators using adaptive fuzzy sliding mode control," in *Proceedings of the International Conference on Intelligent Robots and Systems*, Pittsburgh, PA, USA, August 1995, pp. 156–61.

[17] H. Huang and D. Ho, "Delay-dependent robust control of uncertain stochastic fuzzy systems with time-varying delay," *IET Control Theory and Applications*, vol. 1, no. 4, pp. 1075–1085, 2007.

[18] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, New Jersey, USA: Prentice-Hall, 2002.

[19] K. Kiriakidis, "Nonlinear modeling by interpolation between linear dynamics and its application in control," *Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 6, pp. 813–824, 2007.

[20] Zs. Lendek, R. Babuška, and B. De Schutter, "Stability bounds for fuzzy estimation and control — Part I: State estimation," May 2010, 2010 IEEE International Conference on Automation, Quality and Testing, Robotics.

[21] J. Löfberg, " YALMIP: a toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004, pp. 284–289. [Online]. Available: http://control.ee.ethz.ch/ joloef/yalmip.php

[22] H. Ohtake, K. Tanaka, and H. Wang, "Fuzzy modeling via sector nonlinearity concept," in *Proceedings of the Joint 9th IFSA World Congress and 20th NAFIPS International Conference*, vol. 1, Vancouver, Canada, July 2001, pp. 127–132.

[23] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.

[24] K. Tanaka, T. Ikeda, and H. Wang, "Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs," *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 2, pp. 250–265, 1998.