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Stability analysis and observer design for decentralized TS fuzzy systems

Zs. Lendek R. Babuška B. De Schutter

***Abstract*—A large class of nonlinear systems can be well approximated by Takagi-Sugeno (TS) fuzzy models, with linear or affine consequents. It is well-known that the stability of these consequent models does not ensure the stability of the overall fuzzy system. Stability conditions developed for TS fuzzy systems in general rely on the feasibility of an associated system of linear matrix inequalities, whose complexity may grow exponentially with the number of rules. We study distributed systems, where the subsystems are represented as TS fuzzy models. For such systems, a centralized analysis is often unfeasible. We analyze the stability of the overall TS system based on the stability of the subsystems and the strength of the interconnection terms. For naturally distributed applications, such as multi-agent systems, when adding new subsystems "on-line", the construction and tuning of a centralized observer is often intractable. Therefore, we also propose a decentralized approach to observer design. Applications of such systems include distributed process control, traffic networks, and economic systems.**

I. Introduction

MANY physical systems, such as power systems, communication networks, economic systems, and traffic networks are composed of interconnections of lower-dimensional subsystems. In recent years, the decentralized analysis and control of large-scale interconnected systems has received much attention. A decentralized control scheme usually alleviates the computational costs associated with the centralized control scheme. Also, if new subsystems can be added online, the control scheme does not have to be redesigned.

In this paper we consider the stability analysis of a decentralized system. This class of systems is very important, as many systems are naturally distributed (e.g., multi-agent systems) or cascaded (e.g., hierarchical large-scale systems). Others, though centralized, may be represented as a collection consisting of distributed subsystems, that are less complex than the original system. Earlier works focused on linear systems [1], [2]. However, most physical systems are nonlinear. For practical applications, the linear analysis/synthesis is in general applicable to linearized models of large-scale systems. The disadvantage is that such systems fail to describe nonlinear systems globally. An accurate approximation of a nonlinear system can only be expected in the vicinity of an equilibrium point. Therefore, the analysis is only valid in a region around the operating point, and the performance decreases over a larger domain. In the past several years, active research has been carried out in controller design based on universal approximators, such as fuzzy systems and neural networks.

A large class of nonlinear systems can be represented by TS fuzzy models [3], which in theory can approximate a general nonlinear system to an arbitrary degree of accuracy [4]. The TS fuzzy model consists of a fuzzy rule base. The rule antecedents partition a given subspace of the model variables into fuzzy regions. The consequent of each rule is usually a linear or affine model, valid locally in the corresponding region.

It is well-known that the stability of these local models does not ensure the stability of the overall fuzzy model. Stability conditions have been derived for TS fuzzy systems, most of them relying on the feasibility of an associated system of linear matrix inequalities (LMI) [5]–[7]. A comprehensive survey on the analysis of fuzzy systems can be found in [8]. Recently, much attention has also been paid to the stability of large-scale fuzzy systems.

While decentralized control has received much attention [12]–[17] in the context of large-scale processes and distributed systems, decentralized state estimation has not been addressed as much as the control problem. For decentralized state estimation, generally an architecture with several sensor nodes is assumed, each with its own computing capabilities. In case of a fully decentralized system, computations are performed locally and communication takes place between any two nodes. Each node shares information with other nodes and computes a local estimate. Several topologies have been proposed, depending on the particular application. In case of large-scale processes [18], [19], the network is generally in a hierarchical form, with several intermediate nodes and one final fusion node. For distributed systems, such as multi-agent societies [20], [21], several fusion nodes are used, which process the data and send the information to the rest of the nodes. Observers for distributed estimation include, but are not limited, to the decentralized Kalman and the Extended Kalman filter [22], the information filter, and several types of particle filters [23], [24].

A generic method for the design of an observer valid for all types of nonlinear systems has not yet been developed. For a general nonlinear system represented by a fuzzy model, well-established methods and algorithms can be used to design and to compute fuzzy observers, making the analysis and design much easier. Several types of observers have been developed for TS fuzzy systems, among which: fuzzy Thau-Luenberger observers [5], [9], reduced-order observers [7], [10], and sliding-mode observers [11]. In general, the design

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methods for observers also lead to an LMI feasibility problem. However, the complexity of the LMI problem grows exponentially with the number of rules and the stability analysis problem eventually becomes intractable for a large number of rules.

The contribution of this paper is twofold. First, we consider the stability analysis of a decentralized system, composed of several subsystems. Each subsystem is represented by a TS fuzzy model. The coupling between the subsystems is realized through their states, i.e., the states of a subsystem may influence the dynamics of another subsystem. We propose sequential stability analysis of the overall TS system based on the stability of the subsystems and the strength of these interconnection terms. Second, this approach is extended to observer design. We design stable fuzzy observers sequentially for the subsystems and analyze the joint stability of these fuzzy observers. While this approach is still conservative, it has the benefit that when adding new subsystems, the already existing observers do not have to be altered.

The structure of the paper is as follows. Section II reviews some results for cascaded fuzzy systems. Section III presents the stability conditions for decentralized TS fuzzy systems. The proposed observer design for decentralized fuzzy systems is presented in Section IV. An example is given in Section V. Finally, Section VI concludes the paper.

II. Preliminaries

The stability conditions for TS fuzzy systems generally depend on the feasibility of an associated LMI problem. Since our results rely on stability conditions for cascaded TS systems, some of the relevant stability conditions for this class of systems are reviewed below. Throughout the paper it is assumed that the membership functions are normalized.

A. Cascaded Fuzzy Systems

Consider the autonomous fuzzy system expressed as:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i \boldsymbol{x} \tag{1}$$

where A_i, $i = 1, 2, \ldots, m$ represents the ith local linear model, w_i is the corresponding normalized membership function, and $\boldsymbol{z}$ a vector of scheduling variables. System (1) can also be written as:

$$\dot{\boldsymbol{x}} = A(\boldsymbol{z})\boldsymbol{x} \tag{2}$$

with $A(\boldsymbol{z}) = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i$.

For system (1), several stability conditions have been derived. Among them, a well-known and frequently used condition is formulated as follows [5].

Theorem 1: System (1) *is exponentially stable if there exists* $P = P^T > 0$ *so that* $A_i^T P + P A_i^T < 0$*, for* $i = 1, 2, \ldots, m$.

In what follows, two subsystems are considered, without loss of generality. Assume that the system matrices of the model (1) for each rule $i = 1, 2, \ldots, m$ can be written as:

$$A_i = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}_i = \begin{pmatrix} A_{1i} & 0 \\ A_{21i} & A_{2i} \end{pmatrix}$$

i.e., system (1) can be expressed as a cascade of two fuzzy systems:

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= A_1(\boldsymbol{z}_1)\boldsymbol{x}_1 \\ \dot{\boldsymbol{x}}_2 &= A_{21}(\boldsymbol{z}_1, \boldsymbol{z}_2)\boldsymbol{x}_1 + A_2(\boldsymbol{z}_1, \boldsymbol{z}_2)\boldsymbol{x}_2 \end{aligned} \tag{3}$$

with normalized membership functions w_{1i} and w_{2i}, $\boldsymbol{x} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T]^T$, $\boldsymbol{z} = [\boldsymbol{z}_1^T, \boldsymbol{z}_2^T]^T$, $A_1(\boldsymbol{z}_1) = \sum_{i=1}^{m} w_i(\boldsymbol{z}_1) A_{1i}$, $A_2(\boldsymbol{z}) = \sum_{i=1}^{m} w_{2i}(\boldsymbol{z}) A_{2i}$, etc.

Consider now the subsystems

$$\dot{\boldsymbol{x}}_1 = A_1(\boldsymbol{z}_1)\boldsymbol{x}_1 \tag{4}$$

and

$$\dot{\boldsymbol{x}}_2 = A_2(\boldsymbol{z}_1, \boldsymbol{z}_2)\boldsymbol{x}_2 \tag{5}$$

The following results [25] shows that the stability of the individual subsystems imply the stability of the cascaded system.

Theorem 2: If there exist two Lyapunov functions of the form $V_1(\boldsymbol{x}_1) = \boldsymbol{x}_1^T P_1 \boldsymbol{x}_1$ *and* $V_2(\boldsymbol{x}_2) = \boldsymbol{x}_2^T P_2 \boldsymbol{x}_2$ *so that the subsystems* (4) *and* (5) *are uniformly globally asymptotically stable, then the cascaded system* (3) *is also uniformly globally asymptotically stable.*

Moreover, there exists $\alpha \in \mathcal{R}^+$ so that $V = \boldsymbol{x}^T P \boldsymbol{x}$, with $P = \begin{pmatrix} \alpha P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ is a Lyapunov function for (3).

B. Cascaded Fuzzy Observers

Consider now the affine fuzzy system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned} \tag{6}$$

and an observer of the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}}. \end{aligned} \tag{7}$$

For the purpose of analysis, generally two cases are distinguished: 1) the scheduling vector $\boldsymbol{z}$ does not depend on the estimated states and 2) $\boldsymbol{z}$ depends on (some of) the estimated states, that is $\boldsymbol{z} = \mathbf{f}(\widehat{\boldsymbol{x}})$. In this paper, it is considered that the scheduling vector $\boldsymbol{z}$ does not depend on the estimated states.

Assuming that the system matrices for each rule $i = 1, 2, \ldots, m$ are in cascaded form, observers can be designed individually for each subsystem and each rule, with the overall observer gain having the form $L_i = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}_i = \begin{pmatrix} L_{1i} & 0 \\ 0 & L_{2i} \end{pmatrix}$, where i denotes the rule number.

The cascaded error system can be written as:

$$\begin{aligned} \dot{\boldsymbol{e}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i - L_iC)\boldsymbol{e} \\ &= \sum_{i=1}^{m} w_i(\boldsymbol{z}) \begin{pmatrix} A_{1i} - L_{1i}C_1 & 0 \\ A_{21i} - L_{2i}C_{21} & A_{2i} - L_{2i}C_2 \end{pmatrix} \boldsymbol{e}. \end{aligned} \tag{8}$$

for which the results obtained for cascaded systems can be directly applied.

III. Stability of Decentralized Fuzzy Systems

Suppose that a distributed system is composed of a number of subsystems, each subsystem being represented by a TS fuzzy model. The subsystems are coupled through their states. For the ease of notation and without loss of generality, only two subsystems are considered. The subsystems are therefore expressed as:

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{1i}\boldsymbol{x}_1 + A_{12i}\boldsymbol{x}_2)\\ \dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{2i}\boldsymbol{x}_2 + A_{21i}\boldsymbol{x}_1)\end{aligned} \tag{9}$$

The structure of such a system is presented in Figure 1.

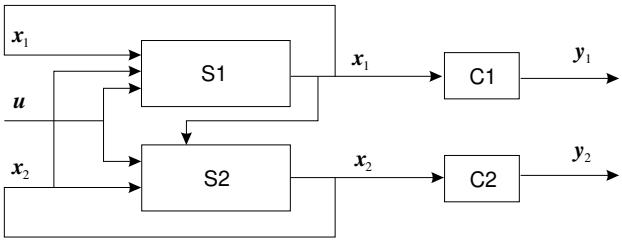

Fig. 1. Two subsystems coupled through their states.

For such a system, the following stability conditions can be formulated:

Theorem 3: The decentralized system (9) *is exponentially stable, if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, *so that*

$$\begin{aligned}&{A_{1_i}}^T P_1 + P_1 A_{1i} < -2Q_1 \quad i = 1, 2, \ldots, m\\ &{A_{2_i}}^T P_2 + P_2 A_{2i} < -2Q_2 \quad i = 1, 2, \ldots, m\\ &\lambda_{\min}(Q_1(\geq \max_i \|P_1 A_{12i}\|\\ &\frac{\lambda_{\min}({A_{1_i}}^T P_1 + P_1 A_{1i} + 2Q_1)}{\max_i \|P_1 A_{12i}\|} >\\ &\qquad \frac{\max_i \|{A_{21_i}}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}({A_{2_i}}^T P_2 + P_2 A_{2i} + 2Q_2)}\end{aligned}$$

where $\lambda_{\min}(.)$ *is the eigenvalue with the smallest absolute magnitude and* $\|\cdot\|$ *denotes the Euclidean norm.*

Remark: If $A_{12i} = 0$ for $i = 1, 2, \ldots, m$ or if $A_{21i} = 0$ for $i = 1, 2, \ldots, m$ then based on Theorem 2, the system (9) is stable if the individual subsystems are stable, and the last two conditions are not needed.

Proof: Consider first the following part of the system (9):

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{1i}\boldsymbol{x}_1)\\ \dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{2i}\boldsymbol{x}_2 + A_{21i}\boldsymbol{x}_1)\end{aligned} \tag{10}$$

This is a cascaded system and it is exponentially stable, if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$ so that

$$\begin{aligned}&{A_{1_i}}^T P_1 + P_1 A_{1i} < -2Q_1 \quad i = 1, 2, \ldots, m\\ &{A_{2_i}}^T P_2 + P_2 A_{2i} < -2Q_2 \quad i = 1, 2, \ldots, m\end{aligned}$$

Then, there exists $\alpha \in \mathcal{R}^+$ so that $V_r = \boldsymbol{x}^T \mathrm{diag}(\alpha P_1,\ P_2)\boldsymbol{x}$ is a Lyapunov function for (10) and $\dot{V}_r < -2\boldsymbol{x}^T Q\boldsymbol{x}$, with $Q = \mathrm{diag}(\alpha Q_1,\ Q_2)$:

$$\dot{V}_r = \sum_{i=1}^{m} w_i(\boldsymbol{z})\boldsymbol{x}^T \begin{pmatrix} \alpha({A_{1_i}}^T P_1 + P_1 A_{1i}) & {A_{21_i}}^T P_2 \\ P_2 A_{21i} & {A_{2_i}}^T P_2 + P_2 A_{2i} \end{pmatrix} \boldsymbol{x}$$

For $\dot{V}_r < -2\boldsymbol{x}^T Q\boldsymbol{x}$, it is needed that

$$\begin{pmatrix} \alpha({A_{1_i}}^T P_1 + P_1 A_{1i}) & {A_{21_i}}^T P_2 \\ P_2 A_{21i} & {A_{2_i}}^T P_2 + P_2 A_{2i} \end{pmatrix} < -2 \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha({A_{1_i}}^T P_1 + P_1 A_{1i} + 2Q_1) & {A_{21_i}}^T P_2 \\ P_2 A_{21i} & {A_{2_i}}^T P_2 + P_2 A_{2i} + 2Q_2 \end{pmatrix} < 0$$

Using the Schur complement, we have

$$\begin{aligned}&\alpha({A_{1_i}}^T P_1 + P_1 A_{1i} + 2Q_1) -\\ &\quad ({A_{21_i}}^T P_2)({A_{2_i}}^T P_2 + P_2 A_{2i} + 2Q_2)^{-1} P_2 A_{21i} < 0\end{aligned}$$

which can be satisfied by choosing α such that

$$\alpha > \frac{\max_i \|{A_{21_i}}^T P_2\|^2}{\lambda_{\min}({A_{1_i}}^T P_1 + P_1 A_{1i} + 2Q_1)\lambda_{\min}({A_{2_i}}^T P_2 + P_2 A_{2i} + 2Q_2)} \tag{11}$$

Now, consider the full system (9). By using the above constructed V_r as a Lyapunov function for (9), we obtain:

$$\begin{aligned}&\dot{V}_r\\ &= \sum_{i=1}^{m} w_i(\boldsymbol{z})\boldsymbol{x}^T \Big(\begin{pmatrix} \alpha({A_{1_i}}^T P_1 + P_1 A_{1i}) & {A_{21_i}}^T P_2 \\ P_2 A_{21i} & {A_{2_i}}^T P_2 + P_2 A_{2i} \end{pmatrix}\\ &\qquad + \begin{pmatrix} 0 & \alpha P_1 A_{12i} \\ \alpha A_{12i} P_1 & 0 \end{pmatrix} \Big)\boldsymbol{x}\\ &< -2\boldsymbol{x}^T \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \boldsymbol{x} + 2\boldsymbol{x}^T \alpha \max_i \|P_1 A_{12i}\| \boldsymbol{x}\\ &< -2\boldsymbol{x}^T \begin{pmatrix} \begin{pmatrix} \alpha(Q_1 - \\ \max_i \|P_1 A_{12i}\| I) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} Q_2 - \\ \alpha \max_i \|P_1 A_{12i}\| I \end{pmatrix} \end{pmatrix} \boldsymbol{x}\end{aligned}$$

which leads to the conditions

$$\lambda_{\min}(Q_1) > \max_i \|P_1 A_{12i}\| \tag{12}$$

$$\lambda_{\min}(Q_2) > \alpha \max_i \|P_1 A_{12i}\| \tag{13}$$

Combining (11) and (13), we get that such an α exists iff

$$\frac{\lambda_{\min}(Q_2)}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|{A_{21_i}}^T P_2\|^2}{\lambda_{\min}({A_{1_i}}^T P_1 + P_1 A_{1i} + 2Q_1)\lambda_{\min}({A_{2_i}}^T P_2 + P_2 A_{2i} + 2Q_2)}$$

or

$$\frac{\lambda_{\min}({A_{1_i}}^T P_1 + P_1 A_{1i} + 2Q_1)}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|{A_{21_i}}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}({A_{2_i}}^T P_2 + P_2 A_{2i} + 2Q_2)} \square$$

The last condition of Theorem 3 couples the subsystems, and therefore the analysis. However, if $c_k = \max_i \|A_{ki}\|$ is known beforehand, the analysis of the subsystems can be decoupled by imposing for all subsystems the conditions:

$$\lambda_{\min} Q_k \geq c_k \|P_k\|$$
$$\lambda_{\min}({A_{k_i}}^T P_k + P_k A_{ki} + 2Q_k) \geq \beta c_k \|P_k\|$$

where $\beta > 1$ is an arbitrary constant and k is the number of the subsystem.

IV. Decentralized Observer Design

Consider now the distributed fuzzy system:

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + A_{12i}\boldsymbol{x}_2)\\ \boldsymbol{y}_1 &= C_1\boldsymbol{x}_1\end{aligned} \tag{14}$$
$$\begin{aligned}\dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1)\\ \boldsymbol{y}_2 &= C_2\boldsymbol{x}_2\end{aligned}$$

for which a decentralized observer has to be designed. It is assumed that the membership functions are normalized and the scheduling vector does not depend on the states to be estimated. The observer considered is of the form:

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}}_1 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{1i}\widehat{\boldsymbol{x}}_1 + B_{1i}\boldsymbol{u} + A_{12i}\widehat{\boldsymbol{x}}_2 + L_{1i}(\boldsymbol{y}_1 - \widehat{\boldsymbol{y}}_1))\\ \widehat{\boldsymbol{y}}_1 &= C_1\widehat{\boldsymbol{x}}_1\end{aligned}$$

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}}_2 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{2i}\widehat{\boldsymbol{x}}_2 + B_{2i}\boldsymbol{u} + A_{21i}\widehat{\boldsymbol{x}}_1 + L_{2i}(\boldsymbol{y}_2 - \widehat{\boldsymbol{y}}_2))\\ \widehat{\boldsymbol{y}}_2 &= C_2\widehat{\boldsymbol{x}}_2\end{aligned} \tag{15}$$

The goal is to design the observer gains L_{1i}, L_{2i}, $i = 1, 2, \ldots, m$ for each subsystem and rule, so that (15) is a stable observer. Even though the observer gains are designed individually for the subsystems, the subsystems are coupled. Contrary to the stabilization problem, when designing observers for distributed systems it is necessary that the (estimated) states of a subsystem that influences another are available to the subsystem that is being influenced. In this sense the design is not fully decentralized. The observer structure is depicted in Figure 2.

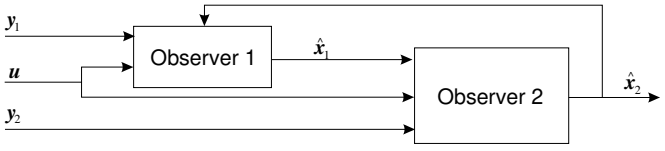

Fig. 2. Decentralized observers for two subsystems.

The error systems can be expressed as:

$$\begin{aligned}\dot{\boldsymbol{e}}_1 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})[A_{1i}\boldsymbol{e}_1 - L_{1i}C_1\boldsymbol{e}_1 + A_{12i}\boldsymbol{e}_2]\\ \boldsymbol{e}_{y1} &= C_1\boldsymbol{e}_1\end{aligned}$$
$$\begin{aligned}\dot{\boldsymbol{e}}_2 &= \sum_{i=1}^{m} w_i(\boldsymbol{z})[A_{2i}\boldsymbol{e}_2 - L_{2i}C_2\boldsymbol{e}_2 + A_{21i}\boldsymbol{e}_1]\\ \boldsymbol{e}_{y2} &= C_2\boldsymbol{e}_2\end{aligned} \tag{16}$$

or

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m} w_i(\boldsymbol{z}) \begin{bmatrix} A_1 - L_1C_1 & A_{12} \\ A_{21} & A_2 - L_2C_2 \end{bmatrix}_i \boldsymbol{e} \tag{17}$$

Using the results from Section III, it can be stated that:

Corollary 1: The error system (17) *is exponentially stable, if there exist* L_{1i}, L_{2i}, $i = 1, 2, \ldots, m$, $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, *so that*

$${G_{1_i}}^T P_1 + P_1 G_{1i} < -2Q_1 \quad i = 1, 2, \ldots, m$$
$${G_{2_i}}^T P_2 + P_2 G_{2i} < -2Q_2 \quad i = 1, 2, \ldots, m$$
$$\lambda_{\min}(Q_1) \geq \max_i \|P_1 A_{12i}\|$$
$$\frac{\lambda_{\min}({G_{1_i}}^T P_1 + P_1 G_{1i} + 2Q_1)}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|{A_{21_i}}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}({G_{2_i}}^T P_2 + P_2 G_{2i} + 2Q_2)}$$

where $G_{1i} = (A_1 - L_1 C)_i$ *and* $G_{2i} = (A_2 - L_2 C)_i$.

Remark: If A_{12i}, $i = 1, 2, \ldots, m$ or A_{21i}, $i = 1, 2, \ldots, m$ are zero, then again the last two conditions are not needed.

Applying Corollary 1 to a practical problem may become tedious, as the observers for the subsystems need to be designed sequentially. For instance, for two subsystems, first L_{2i}, $i = 1, 2, \ldots, m$, $P_2 = P_2^T > 0$, $Q_2 = Q_2^T > 0$, need to be determined and afterwards L_{1i}, $i = 1, 2, \ldots, m$, so that the conditions of Corollary 1 are satisfied. However, when new subsystems are added, such a sequential design is an advantage as the observers for the already existing subsystems do not need to be redesigned.

For two subsystems, the design can be decoupled by analyzing the last condition of Corollary 1. Since $\lambda_{\min}(Q_2)$ can vary between 0 and $\lambda_{\min}({G_{2_i}}^T P_2 + P_2 G_{2i})/2$, the expression

$$\lambda_{\min} Q_2 \lambda_{\min}({G_{2_i}}^T P_2 + P_2 G_{2i})$$

is maximized at $\lambda_{\min}(Q_2^*) = \lambda_{\min}({G_{2_i}}^T P_2 + P_2 G_{2i})/4$ and if there exists such a Q_2^*, we obtain

$$\frac{\max_i \|{A_{21_i}}^T P_2\|^2}{\lambda_{\min}(Q_2^*)\lambda_{\min}({G_{2_i}}^T P_2 + P_2 G_{2i} + 2Q_2^*)} \leq \frac{\max_i \|{A_{21_i}}^T P_2\|^2}{2\lambda_{\min}^2(Q_2^*)}.$$

Second, if $\lambda_{\min}({G_{1_i}}^T P_1 + P_1 G_{1i}) < (2+\gamma)\lambda_{\min}(Q_1)$ for some $\gamma > 0$, then

$$\frac{\lambda_{\min}({G_{1_i}}^T P_1 + P_1 G_{1i} + 2Q_1)}{\max_i \|P_1 A_{12i}\|} > \gamma.$$

By choosing $\gamma = 1/2$, conservativeness is introduced, and the condition $\lambda_{\min}(Q_2) > \max_i \|{A_{21}}_i^T P_2\|$ is obtained. To have the same design for each subsystem, the first condition Corollary 1 should be restricted to ${G_{1}}_i^T P_1 + P_1 G_{1i} < -4Q_1$. Then, the stability conditions can be summarized as follows:

Corollary 2: The error system (17) *is exponentially stable, if there exist* L_{ki}, $i = 1, 2, \ldots, m$, $P_k = P_k^T > 0$, $Q_k = Q_k^T > 0$ *so that*

$$
\begin{aligned}
&{G_{ki}}^T P_k + P_k G_{ki} < -4Q_k \quad i = 1, 2, \ldots, m \\
&\lambda_{\min}(Q_k) \geq \max_i \|P_k A_{ki}\|
\end{aligned}
$$

where $G_{ki} = (A_k - L_k C)_i$ *and* k *is the index of the subsystem.*

V. Example

Here we give a numerical example to illustrate the decentralized observer design. Consider a decentralized system, composed of two subsystems, as follows:

1) Subsystem 1:
   Rule 1: If z_1 is small then

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \begin{pmatrix} -2 & 3 \\ 1.5 & -2.2 \end{pmatrix} \boldsymbol{x}_1 + \begin{pmatrix} 0.08 & 0.05 \\ 0.08 & 0.05 \end{pmatrix} \boldsymbol{x}_2 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \\
y_1 &= (1 \;\; 0)\boldsymbol{x}_1
\end{aligned}
$$

   Rule 2: If z_1 is big then

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \begin{pmatrix} -3 & 1 \\ 5 & -3 \end{pmatrix} \boldsymbol{x}_1 + \begin{pmatrix} 0.1 & 0.06 \\ 0.09 & 0.2 \end{pmatrix} \boldsymbol{x}_2 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u \\
y_1 &= (1 \;\; 0)\boldsymbol{x}_1
\end{aligned}
$$

2) Subsystem 2:
   Rule 1: If z_2 is small then

$$
\begin{aligned}
\dot{\boldsymbol{x}}_2 &= \begin{pmatrix} -3 & 1 \\ 5 & -3 \end{pmatrix} \boldsymbol{x}_2 + \begin{pmatrix} 0.2 & 0.12 \\ 0.2 & 0.12 \end{pmatrix} \boldsymbol{x}_1 + \begin{pmatrix} 3 \\ 4 \end{pmatrix} u \\
y_2 &= (1 \;\; 0)\boldsymbol{x}_2
\end{aligned}
$$

   Rule 2: If z_2 is big then

$$
\begin{aligned}
\dot{\boldsymbol{x}}_2 &= \begin{pmatrix} -2 & 1 \\ 3 & -0.3 \end{pmatrix} \boldsymbol{x}_2 + \begin{pmatrix} 0.1 & 0.36 \\ 0.06 & 0.36 \end{pmatrix} \boldsymbol{x}_1 + \begin{pmatrix} 3 \\ 4 \end{pmatrix} u \\
y_2 &= (1 \;\; 0)\boldsymbol{x}_2
\end{aligned}
$$

The scheduling vectors z_1 and z_2 are independent, but with the same membership functions (see Figure 3). The observers

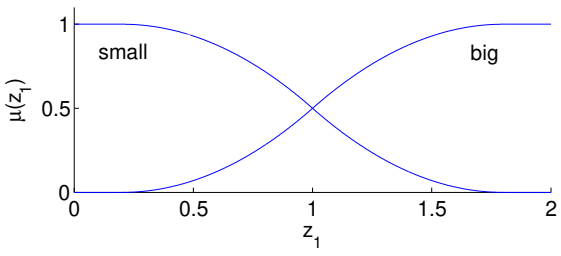

Fig. 3. Membership function of z_1.

were designed independently, using the conditions of Corollary 2, and based on the assumption that for each subsystem the maximum of the norm of the interconnection terms is known beforehand:

$$
\begin{aligned}
c_1 &= \max\left\{ \left\| \begin{pmatrix} 0.08 & 0.05 \\ 0.08 & 0.05 \end{pmatrix} \right\|, \left\| \begin{pmatrix} 0.1 & 0.06 \\ 0.09 & 0.2 \end{pmatrix} \right\| \right\} = 0.24 \\
c_2 &= \max\left\{ \left\| \begin{pmatrix} 0.2 & 0.12 \\ 0.2 & 0.12 \end{pmatrix} \right\|, \left\| \begin{pmatrix} 0.1 & 0.36 \\ 0.06 & 0.36 \end{pmatrix} \right\| \right\} = 0.52
\end{aligned}
$$

Then, the LMIs

$$
\begin{aligned}
&G_{ki}^T P_k + P_k G_{ki} < -4Q_k \\
&\begin{pmatrix} 4(Q_k - c_k^2) & P_k \\ P_k & I \end{pmatrix} < 0
\end{aligned}
$$

$k = 1, 2$, $i = 1, 2$ are solved, obtaining $L_{11} = \begin{pmatrix} 4.89 \\ 5.75 \end{pmatrix}$, $L_{12} = \begin{pmatrix} 2.89 \\ 7.25 \end{pmatrix}$, $L_{21} = \begin{pmatrix} 8.89 \\ 48.34 \end{pmatrix}$, and $L_{22} = \begin{pmatrix} 9.54 \\ 41.89 \end{pmatrix}$.

A typical error trajectory can be seen in Figure 4. For simulation purposes, the system was discretized using the Euler method and a sampling period of $T = 0.05$ s. This particular trajectory was computed for a randomly generated input and scheduling vector, with the true initial state being $[0.1 \; 1 \; 3 \; 0]^T$ and the estimated initial state being $[1 \; 0.2 \; 0 \; 6]^T$. As expected, the error converges asymptotically to zero. A centralized system was obtained by taking all possible

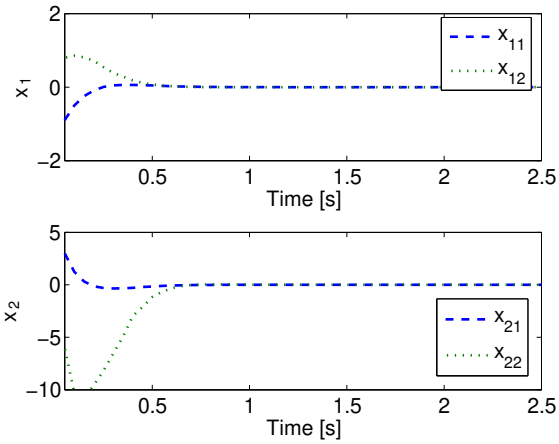

Fig. 4. Error for the two subsystems with decentralized design.

combinations of the subsystems and an observer has been designed for this centralized system. This means that both the number of rules and the dimension of the LMI problem to be solved increases. In the presented case, only 4 LMIs of dimension 4 needs to be solved. However, the number of rules of the centralized system may increase exponentially with the number of subsystems, and therefore the analysis can easily become unfeasible.

The error trajectory, obtained by using the same simulation conditions as for the decentralized system can be seen in Figure 5.

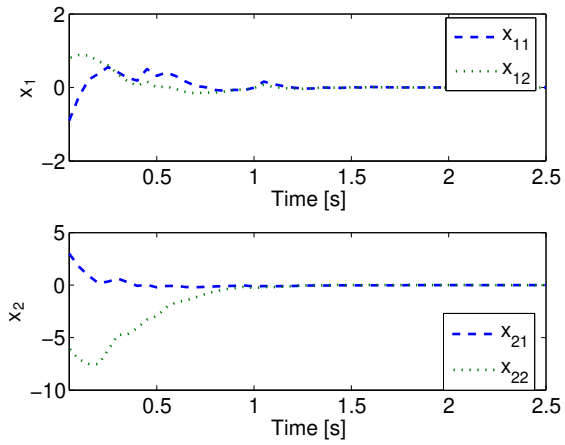

Fig. 5. Error for the two subsystems with centralized design.

VI. Conclusions

Many physical systems, such as power systems, communication networks, economic systems and traffic networks are composed of interconnections of lower-dimensional subsystems. In this paper, the stability of such decentralized systems was studied for the case when the subsystems are represented as TS fuzzy systems. The proposed approach reduces the dimension of the problem to be solved, by analyzing the stability of the overall system based on the individual subsystems and the strength of the interconnection terms. We have also extended this setting to state estimation. Observers can be designed for the individual subsystems sequentially. This partitioning of a process and observer leads to an increased modularity and a reduced complexity of the problem. The benefits of the proposed approach have been demonstrated on a simulation example.

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